格子フェルミオン

2024. 09. 11 @ 格子場の理論夏の学校

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講義の目標

『格子正則化は，QED, QCDのようなVector-likeなゲージ理論に限らず,
SMやGUTようなchiralなゲージ理論に対してさえもゲージ不変な正則化を与え,
摂動的にも非摂動的にも，場の量子論の強力な解析手法を提供してくれる,
おそらく最も発展した正則化法である．』

と主張したい.

この主張の理解にむけて，格子フェルミオンの基礎的な事項を解説したい.

1. 格子フェルミオンの基礎

- Species doubling problem

  Nielsen-Ninomiya定理, Wilsonフェルミオン, Kogut-Susskindフェルミオン

- Positivity

  Transfer Matrix, Functional determinant

- Chiral property of Wilson-Diracフェルミオン

  Axial WT id., Self-energy correction, Chiral anomaly

- Ginsparg-Wilson関係式

  Block-spin変換, Fixed point作用

- Domain-wallフェルミオン

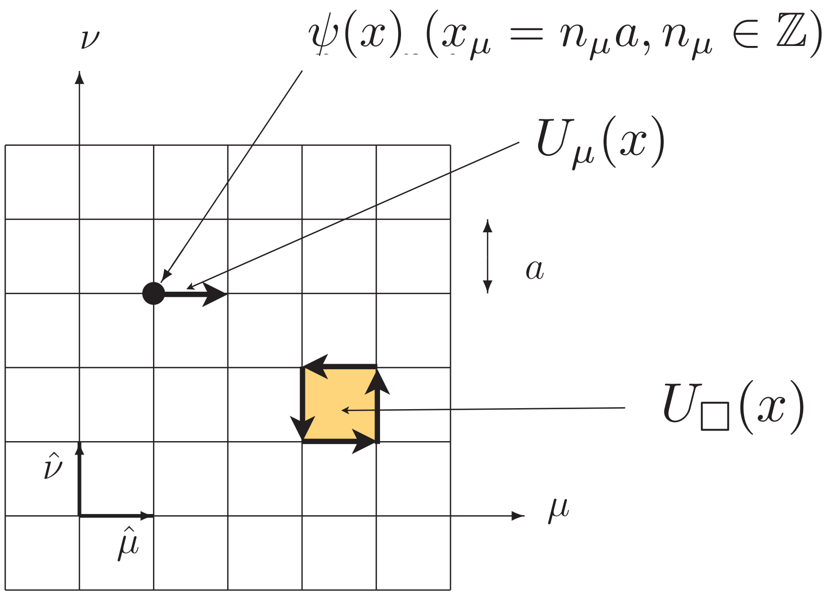
  Chiral edge mode の相関関数, 有効作用

- Overlapフェルミオン

  Locality, Chiral symmetry, Chiral anomaly, Index定理, Topological charge

Lattice Gauge Theories

- d dim. Euclidean lattice with the lattice spacing "*a*"; $x_\mu = n_\mu a$
- Matter fields on the sites, Gauge fields on the links; $\psi(x)$, $U(x, \mu)$



- gauge tr. is defined on the sites

$$\psi(x) \longrightarrow g(x)\psi(x) \quad g(x) \in G$$

$$U_\mu(x) \rightarrow g(x)U_\mu(x)g^{-1}(x+\hat\mu) \quad U_\mu(x) \in G$$

- gauge covariant difference ops. can be defined with the Link field

$$\nabla_\mu\psi(x) = \frac{1}{a}\left(U_\mu(x)\psi(x+\hat\mu a) - \psi(x)\right)$$

Lattice Gauge Theories

- the commutator of the covariant diff. ops. gives the field strength of the gauge link field as plaquette variables

$$[\nabla_\mu, \nabla_\nu]\psi(x) = \Big(1 - U_\Box(x)\Big) U_\mu(x) U_\nu(x + \hat{\mu}a)\psi(x + \hat{\mu}a + \hat{\nu}a)$$

$$U_\Box(x) = U_\mu(x) U_\nu(x + \hat{\mu}a) U_\mu(x + \hat{\nu}a)^{-1} U_\nu(x)^{-1}$$

- Gauge field action can be defined with the plaquette variables [*Wilson(1974)*]

$$S_G = \frac{1}{g^2} \sum_{x\mu\nu} \mathrm{ReTr}\left(1 - U_\Box(x)\right)$$

- Path integral can be defined with the group invariant measure

$$\mathcal{D}[U_\mu(x)] = \prod_{x,\mu} dU_\mu(x)$$

  - Gauge invariance is maintained exactly

- Continuum limit is defined near a 2nd order critical point

$$\xi = \frac{1}{m_{\text{phys}}\, a} \nearrow \infty \qquad m_{\text{phys}} \ll \Lambda = \frac{1}{a} \quad (\nearrow \infty)$$

  - The universal scaling property at the 2nd order PT gives a non-perturbative renormalization procedure.

I. 格子フェルミオンの基礎

Lattice Fermions, Species doubling problem

- Fermion fields on the lattice suffer from the species doubling problem
  - Dirac eq.

$$\mathcal{H} = \sum_{k=1}^{3} \alpha_k \frac{1}{i}\frac{\partial}{\partial x_k} + \beta m \Longrightarrow \mathcal{H}_{\rm lat} = \sum_{k=1}^{3} \alpha_k \frac{1}{2i}\left(\partial_k - \partial_k^\dagger\right) + \beta m$$

$$\partial_k \psi(\boldsymbol{x}, t) = \left(\psi(\boldsymbol{x} + \hat{k}a, t) - \psi(\boldsymbol{x}, t)\right)/a$$

  - Eigenvalue of H_{lat}

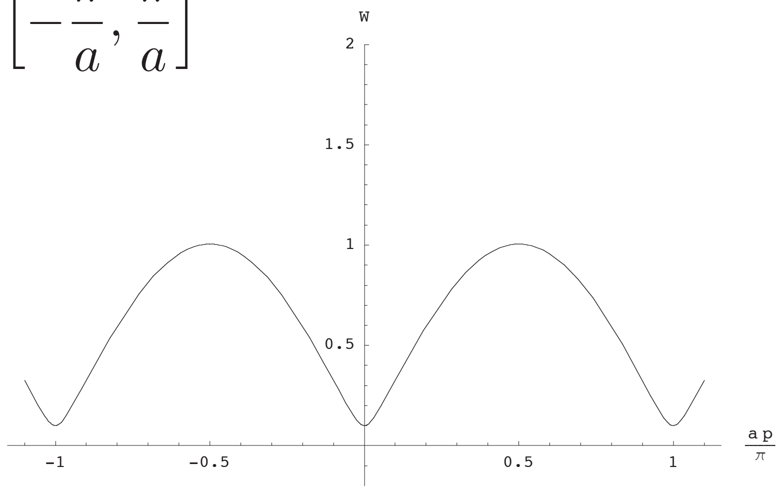
$$E = \pm\sqrt{\sum_{k=1}^{3} \frac{1}{a^2}\sin^2(p_k a) + m^2}, \quad p_1, p_2, p_3 \in \left[-\frac{\pi}{a}, \frac{\pi}{a}\right]$$

  - Species doubling

$$p_k = \pi/a + q_k, \quad |q_k| \ll \pi/a$$

$$\alpha_k \sin(p_k a) \simeq (-\alpha_k) q_k$$

$$\gamma_5 = (-i)\alpha_1\alpha_2\alpha_3 \Rightarrow (-1)^n \times (-i)\alpha_1\alpha_2\alpha_3$$



Lattice Fermions, Species doubling problem

- Nielsen-Ninomiya theorem states that it is a general & inevitable result

[*Nielsen-Ninomiya(1981)*]
[*D.Friedan(1982)*]

$$H = \int d\mathbf{p}\,\psi_a^*(\mathbf{p})K_b^a(\mathbf{p})\psi^b(\mathbf{p})$$

$K(\mathbf{p})$ is an $N \times N$ hermitian matrix which depends smoothly on $\mathbf{p}$

$$[\psi^a(\mathbf{p}), \psi^b(\mathbf{q})]_+ = 0,$$

$$[\psi_b^*(\mathbf{p}), \psi^a(\mathbf{q})]_+ = \delta_b^a \delta(\mathbf{p}-\mathbf{q}).$$

(1) it is quadratic in the fields;
(2) it is invariant under change of the phase of the fields;
(3) it is invariant under translations of the (cubic) lattice; and
(4) it is local, specifically in the sense that it is continuous in momentum space.

$$UKU^{-1}(\mathbf{p}) \sim \left(\begin{array}{c|c} (\mathbf{p}-\mathbf{p}_\alpha)^i \Gamma_i^\alpha & 0 \\ \hline 0 & K_\alpha(\mathbf{p}-\mathbf{p}_\alpha) \end{array}\right)$$

$K_\alpha(0)$ has only nonzero eigenvalues

$$\Gamma_i^\alpha \Gamma_j^\alpha + \Gamma_j^\alpha \Gamma_i^\alpha = 2\delta_{ij}$$

an N_α-dimensional representation of the Clifford algebra on d generators

$$\Gamma_5^{Q,\alpha} = \frac{(-i)^{(d-1)/2}}{d!} \varepsilon^{i_1 \ldots i_d} \Gamma_{i_1}^{Q,\alpha} \ldots \Gamma_{i_d}^{Q,\alpha} \qquad I_{Q,\alpha} = 2^{-(d-1)/2} \operatorname{tr}(\Gamma_5^{Q,\alpha})$$

$$I_Q = \sum_\alpha I_{Q,\alpha} = 0$$

$$\psi^a(\mathbf{p}) \to T^a_b(\mathbf{p})\psi^b(\mathbf{p}) \qquad T(\mathbf{p}) \text{ is unitary}$$

irreducible representation Q

$$P_Q(\mathbf{p})\psi_Q(\mathbf{p}) = \psi_Q(\mathbf{p}) \qquad P_Q(\mathbf{p}) \text{ should depend smoothly on } \mathbf{p}$$

$$[K(\mathbf{p}), P_Q(\mathbf{p})] = 0$$

$$S(p) = P_Q(\mathbf{p})\theta(K(\mathbf{p}) - p^0)$$

$$j^\mu_Q(p) = \left[d\left(\frac{d-1}{2}\right)!(2\pi i)^{(d-1)/2} \right]^{-1} \varepsilon^{\mu\nu_1\ldots\nu_d}\operatorname{tr}(\partial_{\nu_1}S\ldots\partial_{\nu_d}S)(p)$$

(P1) $\partial_\mu j^\mu_Q(p) = 0$, i.e. chiral charge is conserved.
(P2) $j^\mu_Q(p)$ vanishes away from the spectrum of K;
(P3) $j^\mu_Q(p)$ depends only on the spectral projection of K near p;
(P4) $j^0_Q(0, \mathbf{p}) = \sum_\alpha \delta(\mathbf{p} - \mathbf{p}_\alpha) I_{Q.\alpha}$,

$$I_Q = \int d\mathbf{p}\, j^0_Q(0, \mathbf{p})$$

Lattice Fermions, Species doubling problem

- Nielsen-Ninomiya theorem states that it is a general & inevitable result

[*Nielsen-Ninomiya(1981)*]

[*Karsten (1981)*]

[*Luscher (1998)*]

$$S = a^4 \sum_x \bar{\psi}(x)\, D\, \psi(x) = \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4}\, \bar{\psi}(-k)\, \tilde{D}(k)\, \psi(k)$$

1. $\tilde{D}(k)$ is a periodic and analytic function of momentum k_μ
2. $\tilde{D}(k) \propto i\gamma_\mu k_\mu \quad \text{for} \quad |k_\mu| \ll \pi/a$
3. $\tilde{D}(k)$ is invertible for all k_μ except $k_\mu = 0$
4. $\gamma_5 \tilde{D}(k) + \tilde{D}(k)\gamma_5 = 0$

smoothness, analyticity & locality :

$$\frac{\partial^l}{\partial k^l} \tilde{D}(k) = \sum_x e^{ikx} (ix)^l D(x) < \infty \quad \Longrightarrow \quad \|D(x)\| < C e^{-\gamma|x|}$$

example :

$$\tilde{D}(k) = i\gamma_\mu \mathrm{e}^{ip_\mu/2}\, 2\sin(p_\mu/2)$$

$$\tilde{D}(k) = \gamma_\mu P_+ i F_\mu^{(+)}(k) + \gamma_\mu P_- i F_\mu^{(-)}(k)$$ T^3 上のベクトル場

$$F_\mu^{(+)}(k_0^{\alpha_+}) = 0 \qquad F_\mu^{(-)}(k_0^{\alpha_-}) = 0$$

$$\mathrm{I} = \sum_{k=k^{\alpha_+}} \frac{\det\left\{\partial_\nu F_\mu^{(+)}(k^{\alpha_+})\right\}}{\left|\det\left\{\partial_\nu F_\mu^{(+)}(k^{\alpha_+})\right\}\right|} - \sum_{k=k^{\alpha_-}} \frac{\det\left\{\partial_\nu F_\mu^{(-)}(k^{\alpha_-})\right\}}{\left|\det\left\{\partial_\nu F_\mu^{(-)}(k^{\alpha_-})\right\}\right|}$$

ポワンカレ-ホップ定理より

$V_\mu(k)$ T^3 上のベクトル場

$V_\mu(k^\alpha) = 0$ 孤立したゼロ点 $\alpha = 1, \cdots$

$$I = \sum_\alpha \frac{\det\{\partial_\nu V_\mu(k^\alpha)\}}{|\det\{\partial_\nu V_\mu(k^\alpha)\}|} = 0 \quad (= \chi)$$

Lattice Fermions

- Wilson term resolves the degeneracy of doublers, but breaks chiral symm.

$$S_{\rm w} = a^4 \sum_x \bar{\psi}(x) \left(\gamma_\mu \frac{1}{2} \left(\nabla_\mu - \nabla_\mu^\dagger \right) + \boxed{\frac{a}{2} \left(\nabla_\mu \nabla_\mu^\dagger \right)} + m_0 \right) \psi(x)$$

$$m_0 + \sum_\mu \frac{a}{2} \left(\frac{2}{a} \sin \frac{k_\mu a}{2} \right)^2 \simeq m_0 + \frac{2n}{a} \qquad n = \text{numbers of } \pi$$

- Kogut-Suskind single-component fermion can form four-component Dirac fermions, but with four flavors, on the blocked-lattice

$$S_{\rm KS} = \sum_{n,\mu} \frac{1}{2} \eta_\mu(n) \, \bar{\chi}(n) \, \left[\chi(n+\hat{\mu}) - \chi(n-\hat{\mu}) \right]$$

$$\eta_\mu(n) = (-1)^{n_1 + \cdots + n_{\mu-1}} , \quad \eta_1 = 1 \qquad T(n)\gamma_\mu T^\dagger(n+\hat{\mu}) = \eta_\mu(n) \, \boldsymbol{I}$$

$$n_\mu = 2x_\mu + \rho_\mu \qquad T(n) = \gamma_1^{n_1} \gamma_2^{n_2} \gamma_3^{n_3} \gamma_4^{n_4}$$

$$S_{\rm KS} = \sum_{x,\mu} \bar{\psi}(x) \left[(\gamma_\mu \otimes \mathbb{I}) \frac{1}{2} (\partial_\mu - \partial_\mu{}^\dagger) - \frac{1}{2} (\gamma_5 \otimes \gamma_\mu^* \gamma_5^*) \partial_\mu \partial_\mu{}^\dagger \right] \psi(x)$$

$$\psi(x) \to e^{i\alpha(\gamma^5 \otimes \gamma^{5*})} \, \psi(x)$$

$$\bar{\psi}(x) \to \bar{\psi}(x) \, e^{i\alpha(\gamma^5 \otimes \gamma^{5*})}$$

Positivity — Partition function, Transfer-matrix, Functional determinant

$$Z_{\mathrm{W}} = \int \mathcal{D}\bar{\psi}\mathcal{D}\bar{\psi}\,\mathrm{e}^{-S_{\mathrm{W}}}$$

$$= \int \prod_x d\bar{\psi}(x) d\psi(x)\,\mathrm{e}^{-a^4 \sum_x \bar{\psi}(x)(D_{\mathrm{w}}+m_0)\psi(x)} = \mathrm{Det}(D_{\mathrm{w}} + m_0)$$

$$= \int \prod_t dx^\dagger dy^\dagger dx dy\,\mathrm{e}^{-(x^\dagger x + y^\dagger y)} \qquad \psi_t = \frac{1}{\sqrt{B_t}} \begin{pmatrix} x_t \\ {y_t}^\dagger \end{pmatrix} \qquad \bar{\psi}_t = \begin{pmatrix} {x_t}^\dagger \\ -y_t \end{pmatrix} \frac{1}{\sqrt{B_t}}$$

$$\prod_{t=T-1,1} \boxed{T_W({x^\dagger}_{t+1}, {y^\dagger}_{t+1}; x_t, y_t)}\, T_W({x^\dagger}_1, {y^\dagger}_1; x_0, y_0)\Big|_{x_0=-x_T, y_0=-y_T}$$

$$= \mathrm{Det}\left(1 + \prod_{t=T}^{1} \boxed{T_t}\right) \cdot \prod_t \mathrm{Det}(P_+ B_t + P_-) \qquad T = \frac{1 - \mathcal{H}/2}{1 + \mathcal{H}/2}$$

$$\mathcal{H} = a_0 \gamma_0 (D_w^{(3)} + m_0) \frac{1}{1 + a_0(D_w^{(3)} + m_0)/2}$$

$a = 1$

$$D_{\mathrm{w}} = \gamma_\mu \frac{1}{2}(\nabla_\mu - {\nabla_\mu}^\dagger) + \frac{1}{2}\nabla_\mu {\nabla_\mu}^\dagger$$

$$= -\left(\frac{1-\gamma_\mu}{2}\right)\nabla_\mu - \left(\frac{1+\gamma_\mu}{2}\right){\nabla_\mu}^\dagger$$

$$a_0 \neq 1 \qquad \gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \gamma_k = \begin{pmatrix} & i\sigma_k \\ -i\sigma & 0 \end{pmatrix}$$

$$a_0 D_{\mathrm{w}} = \begin{pmatrix} B & C \\ -C^\dagger & B \end{pmatrix} \qquad C = a_0\, i\sigma_k \frac{1}{2}(\nabla_k - {\nabla_k}^\dagger) \qquad B = 1 + a_0 \frac{1}{2}\nabla_k {\nabla_k}^\dagger$$

$$a = 1$$

$$\begin{aligned} D_{\mathrm{w}} &= \gamma_\mu \frac{1}{2}(\nabla_\mu - {\nabla_\mu}^\dagger) + \frac{1}{2}\nabla_\mu {\nabla_\mu}^\dagger \\ &= -\left(\frac{1-\gamma_\mu}{2}\right)\nabla_\mu - \left(\frac{1+\gamma_\mu}{2}\right){\nabla_\mu}^\dagger \end{aligned}$$

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

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$$a_0 \neq 1$$

$$a_0 D_{\mathrm{w}} = \begin{pmatrix} B & C \\ -C^\dagger & B \end{pmatrix}$$

$$C = a_0\, i\sigma_k \frac{1}{2}(\nabla_k - {\nabla_k}^\dagger)$$

$$B = 1 + a_0 \frac{1}{2}\nabla_k {\nabla_k}^\dagger$$

$$\psi_t = \frac{1}{\sqrt{B_t}}\begin{pmatrix} x_t \\ {y_t}^\dagger \end{pmatrix} \qquad \bar{\psi}_t = \begin{pmatrix} {x_t}^\dagger \\ -y_t \end{pmatrix}\frac{1}{\sqrt{B_t}}$$

$$D_W\Psi = \begin{bmatrix} B & C & 0 & 0 & & & & & +1 & 0 \\ -C & B & 0 & -1 & & & & & 0 & 0 \\ -1 & 0 & B & C & 0 & 0 & & & & \\ 0 & 0 & -C & B & 0 & -1 & & & & \\ & & -1 & 0 & B & C & 0 & 0 & & \\ & & 0 & 0 & -C & B & 0 & -1 & & \\ & & & & -1 & 0 & B & C & 0 & 0 \\ & & & & 0 & 0 & -C & B & 0 & -1 \\ 0 & 0 & & & & & -1 & 0 & B & C \\ 0 & +1 & & & & & 0 & 0 & -C & B \end{bmatrix} \begin{pmatrix} \psi_{+1} \\ \psi_{-1} \\ \psi_{+2} \\ \psi_{-2} \\ \psi_{+3} \\ \psi_{-3} \\ \psi_{+4} \\ \psi_{-4} \\ \psi_{+5} \\ \psi_{-5} \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} C & B & 0 & 0 & & & & & 0 & +1 \\ B & -C & -1 & 0 & & & & & 0 & 0 \\ 0 & -1 & C & B & 0 & 0 & & & & \\ 0 & 0 & B & -C & -1 & 0 & & & & \\ & & 0 & -1 & C & B & 0 & 0 & & \\ & & 0 & 0 & B & -C & -1 & 0 & & \\ & & & & 0 & -1 & C & B & 0 & 0 \\ & & & & 0 & 0 & B & -C & -1 & 0 \\ 0 & 0 & & & & & & & C & B \\ +1 & 0 & & & & & & & B & -C \end{bmatrix} \begin{pmatrix} \psi_{-1} \\ \psi_{+1} \\ \psi_{-2} \\ \psi_{+2} \\ \psi_{-3} \\ \psi_{+3} \\ \psi_{-4} \\ \psi_{+4} \\ \psi_{-5} \\ \psi_{+5} \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} B & 0 & 0 & & & & & 0 & +1 & C \\ -C & -1 & 0 & & & & & 0 & 0 & B \\ -1 & C & B & 0 & 0 & & & & & 0 \\ 0 & B & -C & -1 & 0 & & & & & 0 \\ & 0 & -1 & C & B & 0 & 0 & & & \\ & 0 & 0 & B & -C & -1 & 0 & & & \\ & & & 0 & -1 & C & B & 0 & 0 & \\ & & & 0 & 0 & B & -C & -1 & 0 & \\ 0 & & & & & 0 & -1 & C & B & 0 \\ 0 & & & & & 0 & 0 & B & -C & +1 \end{bmatrix} \begin{pmatrix} \psi_{+1} \\ \psi_{-2} \\ \psi_{+2} \\ \psi_{-3} \\ \psi_{+3} \\ \psi_{-4} \\ \psi_{+4} \\ \psi_{-5} \\ \psi_{+5} \\ \psi_{-1} \end{pmatrix}$$

$$
D_W \Psi = \begin{bmatrix}
B & C & 0 & 0 & & & & & +1 & 0 \\
-C & B & 0 & -1 & & & & & 0 & 0 \\
-1 & 0 & B & C & 0 & 0 & & & & \\
0 & 0 & -C & B & 0 & -1 & & & & \\
& & -1 & 0 & B & C & 0 & 0 & & \\
& & 0 & 0 & -C & B & 0 & -1 & & \\
& & & & -1 & 0 & B & C & 0 & 0 \\
& & & & 0 & 0 & -C & B & 0 & -1 \\
0 & 0 & & & & & -1 & 0 & B & C \\
0 & +1 & & & & & 0 & 0 & -C & B
\end{bmatrix}
\begin{pmatrix}
\psi_{+1} \\ \psi_{-1} \\ \psi_{+2} \\ \psi_{-2} \\ \psi_{+3} \\ \psi_{-3} \\ \psi_{+4} \\ \psi_{-4} \\ \psi_{+5} \\ \psi_{-5}
\end{pmatrix}
$$

$$
\Rightarrow \begin{pmatrix} x^{\dagger}{}_t \\ -y_{t+1} \end{pmatrix}
\overset{\begin{pmatrix} y^{\dagger}{}_t & x_{t+1} \end{pmatrix}}{\begin{bmatrix}
C & B & 0 & 0 & & & & & 0 & +1 \\
B & -C & -1 & 0 & & & & & 0 & 0 \\
0 & -1 & C & B & 0 & 0 & & & & \\
0 & 0 & B & -C & -1 & 0 & & & & \\
& & 0 & -1 & C & B & 0 & 0 & & \\
& & 0 & 0 & B & -C & -1 & 0 & & \\
& & & & 0 & -1 & C & B & 0 & 0 \\
& & & & 0 & 0 & B & -C & -1 & 0 \\
0 & 0 & & & & & & & C & B \\
+1 & 0 & & & & & & & B & -C
\end{bmatrix}}
\begin{pmatrix}
\psi_{-1} \\ \psi_{+1} \\ \psi_{-2} \\ \psi_{+2} \\ \psi_{-3} \\ \psi_{+3} \\ \psi_{-4} \\ \psi_{+4} \\ \psi_{-5} \\ \psi_{+5}
\end{pmatrix}
$$

$$T_W(x^\dagger{}_{t+1}, y^\dagger{}_{t+1}; x_t, y_t) = \mathrm{e}^{-x^\dagger{}_{t+1}\frac{1}{\sqrt{B}_{t+1}}C_{t+1}\frac{1}{\sqrt{B}_{t+1}}y^\dagger{}_{t+1}}$$

$$\mathrm{e}^{-\left(-x^\dagger{}_{t+1}\frac{1}{\sqrt{B}_{t+1}}\frac{1}{\sqrt{B}_t}x_t + y_t\frac{1}{\sqrt{B}_t}\frac{1}{\sqrt{B}_{t+1}}y^\dagger{}_{t+1}\right)}$$

$$\mathrm{e}^{-y_t\frac{1}{\sqrt{B}_t}C_t\frac{1}{\sqrt{B}_t}x_t} \qquad \det\sqrt{B}_t \det\sqrt{B}_{t+1}$$

$$= \langle x^\dagger y^\dagger | \hat{T}_W | x, y\rangle$$

$$\mathbb{I} = \int dx^\dagger dy^\dagger dx dy\, \mathrm{e}^{-(x^\dagger x + y^\dagger y)} |x, y\rangle\langle x^\dagger y^\dagger|$$

$$\hat{T}_W(U_{t+1}, U_t) = T_F(U_{t+1})^\dagger T_F(U_t)$$

$$T_F(U_t) = \mathrm{e}^{-\chi^\dagger\frac{1}{2}(1-\gamma_0)C\chi}\mathrm{e}^{-\chi^\dagger \ln\sqrt{B}_t\chi}\ \det B_t^{1/4}$$

$$\chi = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \chi^\dagger = \begin{pmatrix} x^\dagger & y^\dagger \end{pmatrix}$$

$$K_T(U,U') = T_F^+(U)\cdot T_G^+(U)\cdot S(U,U')\cdot T_G(U')\cdot T_F(U')$$

$$T_G(U) = \exp(2g_0)^{-2}\sum_{\boldsymbol{n}}\sum_{i\neq j=1,2,3} \mathrm{Tr}\{U(\boldsymbol{n},i)U(\boldsymbol{n}+\hat{\imath},j)U^+(\boldsymbol{n}+\hat{\jmath},i)U^+(\boldsymbol{n},j)\}$$

$$S(U,U') = \exp\tfrac{1}{2}g_0^{-2}\sum_{\boldsymbol{n}}\sum_{j=1,2,3}\{\mathrm{Tr}[U(\boldsymbol{n},j)U'^+(\boldsymbol{n},j)] + \mathrm{Tr}[U'(\boldsymbol{n},j)U^+(\boldsymbol{n},j)]\}$$

Proposition 1. a) $\hat{T}$ *is a selfadjoint, bounded operator in* $\hat{\mathscr{H}}$.

b) *It is gauge invariant under the restricted class of gauge transformations discussed in the preceding paragraph.*

c) *It is strictly positive, i.e. all its eigenvalues are larger than zero.*

$$S_G = \sum_{\tau,x}\sum_{\mu\nu}\frac{1}{2g^2}\mathrm{Tr}\{(1-U_{\mu\nu})(1-U_{\mu\nu})^\dagger\}$$
$$= \sum_{\tau,x}\sum_{k}\frac{1}{g^2}\mathrm{Tr}\{2-U_k(\tau,x)U_k^\dagger(\tau+1,x)-U_k^\dagger(\tau,x)U_k(\tau+1,x)\}$$
$$+\sum_{\tau,x}\sum_{kl}\frac{1}{2g^2}\mathrm{Tr}\{(1-U_{kl})(1-U_{kl})^\dagger\}(\tau,x)$$

$$T_G(U,U') = \mathrm{e}^{-\sum_x\sum_{kl}\frac{1}{4g^2}\mathrm{Tr}\left\{(1-U_{kl})(1-U_{kl})^\dagger\right\}} \times$$
$$\mathrm{e}^{-\sum_x\sum_k\frac{1}{g^2}\mathrm{Tr}\left\{2-U_kU_k'^\dagger-U_k'U_k^\dagger\right\}} \times$$
$$\mathrm{e}^{-\sum_x\sum_{kl}\frac{1}{4g^2}\mathrm{Tr}\left\{(1-U'_{kl})(1-U'_{kl})^\dagger\right\}}$$
$$= \mathrm{e}^{-\sum_x\sum_{kl}\frac{1}{4g^2}\mathrm{Tr}\left\{(1-U_{kl})(1-U_{kl})^\dagger\right\}} \times$$
$$\prod_x\prod_k \mathrm{e}^{-\frac{2N_c}{g^2}}\sum_r d_r\,L_r\left(2N_c/g^2\right)\mathrm{Tr}_r\{U_kU_k'^\dagger\} \times$$
$$\mathrm{e}^{-\sum_x\sum_{kl}\frac{1}{4g^2}\mathrm{Tr}\left\{(1-U'_{kl})(1-U'_{kl})^\dagger\right\}}.$$

$$\mathrm{e}^{-\frac{1}{g^2}\mathrm{Tr}\left\{2-U-U^\dagger\right\}} = \mathrm{e}^{-\frac{2N_c}{g^2}}\sum_r d_r\,L_r\left(2N_c/g^2\right)\mathrm{Tr}_r\{U\}$$
$$\mathrm{e}^{-\frac{1}{g^2}\left\{1-\cos\theta\right\}} = \mathrm{e}^{-\frac{1}{g^2}}\sum_m I_m\left(1/g^2\right)\mathrm{e}^{im\theta}$$

functions f on the gauge group $G = SU(N)$

$$\int dU dU' f^*(U) \exp \tfrac{1}{2} g_0^{-2} \{\mathrm{Tr}(U^{-1}U') + \mathrm{Tr}(U^{-1}U')^+\} f(U') > 0$$

$$\exp \tfrac{1}{2} g_0^{-2} \{\mathrm{Tr}\, V + \mathrm{Tr}\, V^+\} = \sum_{\nu \in \hat{G}} c_\nu \chi^{(\nu)}(V) \qquad (V = U^{-1}U')$$

$$c_\nu = \sum_{n,m=0}^{\infty} a_{nm} c_\nu(n, m)$$

$$\exp \tfrac{1}{2} g_0^{-2} \{\mathrm{Tr}\, V + \mathrm{Tr}\, V^+\} = \sum_{n,m=0}^{\infty} a_{nm} (\mathrm{Tr}\, V)^n (\mathrm{Tr}\, V^+)^m, \qquad a_{nm} > 0$$

$$(\mathrm{Tr}\, V)^n (\mathrm{Tr}\, V^+)^m = \sum_{\nu \in \hat{G}} c_\nu(n, m) \chi^{(\nu)}(V)$$

the number of times the irreducible representation ν

$\therefore$ c_ν are all positive.

$$
\begin{aligned}
D_W \Psi \;=\;& \begin{bmatrix}
B & C & 0 & 0 & & & & & +1 & 0 \\
-C & B & 0 & -1 & & & & & 0 & 0 \\
-1 & 0 & B & C & 0 & 0 & & & & \\
0 & 0 & -C & B & 0 & -1 & & & & \\
& & -1 & 0 & B & C & 0 & 0 & & \\
& & 0 & 0 & -C & B & 0 & -1 & & \\
& & & & -1 & 0 & B & C & 0 & 0 \\
& & & & 0 & 0 & -C & B & 0 & -1 \\
0 & 0 & & & & & -1 & 0 & B & C \\
0 & +1 & & & & & 0 & 0 & -C & B
\end{bmatrix}
\begin{pmatrix}
\psi_{+1} \\ \psi_{-1} \\ \psi_{+2} \\ \psi_{-2} \\ \psi_{+3} \\ \psi_{-3} \\ \psi_{+4} \\ \psi_{-4} \\ \psi_{+5} \\ \psi_{-5}
\end{pmatrix} \\
\Rightarrow\;& \begin{bmatrix}
C & B & 0 & 0 & & & & & 0 & +1 \\
B & -C & -1 & 0 & & & & & 0 & 0 \\
0 & -1 & C & B & 0 & 0 & & & & \\
0 & 0 & B & -C & -1 & 0 & & & & \\
& & 0 & -1 & C & B & 0 & 0 & & \\
& & 0 & 0 & B & -C & -1 & 0 & & \\
& & & & 0 & -1 & C & B & 0 & 0 \\
& & & & 0 & 0 & B & -C & -1 & 0 \\
0 & 0 & & & & & & & C & B \\
+1 & 0 & & & & & & & B & -C
\end{bmatrix}
\begin{pmatrix}
\psi_{-1} \\ \psi_{+1} \\ \psi_{-2} \\ \psi_{+2} \\ \psi_{-3} \\ \psi_{+3} \\ \psi_{-4} \\ \psi_{+4} \\ \psi_{-5} \\ \psi_{+5}
\end{pmatrix} \\
\Rightarrow\;& \begin{bmatrix}
B & 0 & 0 & & & & & 0 & +1 & C \\
-C & -1 & 0 & & & & & 0 & 0 & B \\
-1 & C & B & 0 & 0 & & & & & 0 \\
0 & B & -C & -1 & 0 & & & & & 0 \\
& 0 & -1 & C & B & 0 & 0 & & & \\
& 0 & 0 & B & -C & -1 & 0 & & & \\
& & & 0 & -1 & C & B & 0 & 0 & \\
& & & 0 & 0 & B & -C & -1 & 0 & \\
0 & & & & & 0 & -1 & C & B & 0 \\
0 & & & & & 0 & 0 & B & -C & +1
\end{bmatrix}
\begin{pmatrix}
\psi_{+1} \\ \psi_{-2} \\ \psi_{+2} \\ \psi_{-3} \\ \psi_{+3} \\ \psi_{-4} \\ \psi_{+4} \\ \psi_{-5} \\ \psi_{+5} \\ \psi_{-1}
\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_t &= \begin{pmatrix} B_t & 0 \\ -C_t & -1 \end{pmatrix}, \qquad \alpha_t^{-1} = \begin{pmatrix} B_t^{-1} & 0 \\ -C_t B_t^{-1} & -1 \end{pmatrix}, \\
\alpha'_N &= \begin{pmatrix} B_N & 0 \\ -C_N & +1 \end{pmatrix} = \alpha_N \gamma_0, \\
\beta_t &= \begin{pmatrix} -1 & C_t \\ 0 & B_t \end{pmatrix}, \qquad \beta_t^{-1} = \begin{pmatrix} -1 & C_t B_t^{-1} \\ 0 & B_t^{-1} \end{pmatrix}, \\
\beta'_1 &= \begin{pmatrix} +1 & C_1 \\ 0 & B_1 \end{pmatrix} = \beta_1(-\gamma_0),
\end{aligned}
$$

$$
\begin{aligned}
\det D_W &= \det\left\{\alpha'_N + \beta_N(-\alpha_{N-1}^{-1})\beta_{N-1}\cdots(-\alpha_1^{-1})\beta'_1\right\}\prod_{i=1}^{N-1}\det\alpha_i \\
&= \det\left\{\alpha_N^{-1}\alpha'_N - (-\alpha_N^{-1})\beta_N(-\alpha_{N-1}^{-1})\beta_{N-1}\cdots(-\alpha_1^{-1})\beta_1\beta_1^{-1}\beta'_1\right\}\prod_{i=1}^{N}\det\alpha_i \\
&= \det\left\{\left(1+\frac{1-\mathcal{H}_N/2}{1+\mathcal{H}_N/2}\frac{1-\mathcal{H}_{N-1}/2}{1+\mathcal{H}_{N-1}/2}\cdots\frac{1-\mathcal{H}_1/2}{1+\mathcal{H}_1/2}\right)\gamma_0\right\}\prod_{i=1}^{N}\det\alpha_i
\end{aligned}
$$

$$
\begin{aligned}
(-\alpha_t^{-1})\beta_t &= \frac{1-\mathcal{H}_t/2}{1+\mathcal{H}_t/2}, \\
\beta_t(-\alpha_{t-1}^{-1}) &= [(2+a_0D_{3w})\gamma_0]_t(1-\mathcal{H}_t/2)\frac{1}{1+\mathcal{H}_{t-1}/2}\frac{1}{[(2+a_0D_{3w})\gamma_0]_{t-1}}
\end{aligned}
$$

$$
\mathcal{H}/2 \equiv \gamma_0 a_0 D_{3w}\frac{1}{2+a_0D_{3w}}
$$

$$
\begin{aligned}
T_F &= B_t^{-1/2}\left[\beta_t(-\alpha_{t-1}^{-1})\right]B_{t-1}^{1/2} \\
&= B_t^{-1/2}[(2+a_0D_{3w})\gamma_0]_t(1-\mathcal{H}_t/2)\frac{1}{1+\mathcal{H}_{t-1}/2}\frac{1}{[(2+a_0D_{3w})\gamma_0]_{t-1}}B_{t-1}^{1/2}
\end{aligned}
$$

$$
\mathcal{H} = a_0\gamma_0(D_w^{(3)}+m_0)\frac{1}{1+a_0(D_w^{(3)}+m_0)/2}
$$

$$
T_F(\Delta) = B_t^{-1/2}[(2+a_0D_{3w})\gamma_0]_t(1-\Delta\mathcal{H}_t/2)\frac{1}{1+\Delta\mathcal{H}_{t-1}/2}\frac{1}{[(2+a_0D_{3w})\gamma_0]_{t-1}}B_{t-1}^{1/2}
$$

$$
T_F(\Delta)T_F(-\Delta) = 1 \qquad \Delta = \pm i
$$

Lattice QCD at finite temperature and Baryon density

$$Z[\beta,\mu] \;=\; \mathrm{Tr}\,\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})} \qquad \beta = N_\beta a_0$$

$$Z_{\text{lat}}[\beta,\mu] \;=\; \int \mathcal{D}[U]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\,\mathrm{e}^{-S_G[U]-S_W[\psi,\bar{\psi},\mu]}$$

$$\begin{aligned} S_W &= \sum_{\tau,x} \bar{\psi}(\tau,x) D_W \psi(\tau,x) \\ &= \sum_{\tau,\tau',x} \bar{\psi}(\tau,x)\Bigg\{-\left(\frac{1-\gamma_0}{2}\right)\delta_{\tau+1,\tau'}\,\mathrm{e}^{-\mu} - \left(\frac{1+\gamma_0}{2}\right)\delta_{\tau,\tau'+1}\,\mathrm{e}^{+\mu} \\ &\qquad + \left(1 + a_0\left[\gamma_k \frac{1}{2}\left(\nabla_k - \nabla_k^\dagger\right) + \frac{1}{2}\nabla_k\nabla_k^\dagger + m\right]\right)\delta_{\tau,\tau'}\Bigg\}\psi(\tau',x) \end{aligned}$$

$$F_{\text{lat}}[\beta,\mu] \;=\; -\frac{1}{\beta}\ln Z_{\text{lat}}[\beta,\mu] \simeq c_0 \frac{1}{a^2}\mu + c_1\mu^3 + \cdots$$

$$U_\mu(\tau,\mathbf{x}) \to U_\mu(\tau,\mathbf{x})\,\mathrm{e}^{-\mu} \qquad c_0 = 0$$

[Hasenfratz and Karsch]

$$\mu = 0 \qquad D_W^\dagger = \gamma_5 D_W \gamma_5 \qquad \det D_W \in \mathbb{R}$$

$$\mu \neq 0 \qquad D_W^\dagger \neq \gamma_5 D_W \gamma_5 \qquad \det D_W \in \mathbb{C} \qquad \{\det D_W(\mu)\}^* = \{\det D_W(-\mu)\}$$

$$\chi_{n_4}(\boldsymbol{n}) \equiv \chi(n)\,, \qquad \chi^\dagger_{n_4}(\boldsymbol{n}) \equiv \frac{1}{2}\,\bar{\chi}(n)\eta_4(n) \qquad \eta'_k(\boldsymbol{n}) \equiv \eta_k(n)\eta_4(n) \qquad \eta'_k(\boldsymbol{n}+\hat{\boldsymbol{k}}) = -\eta'_k(\boldsymbol{n})$$

$$S_F = \sum_{n_4,\boldsymbol{n}} \{\chi^\dagger_{n_4}(\boldsymbol{n})\chi_{n_4+1}(\boldsymbol{n}) + \chi_{n_4}(\boldsymbol{n})\chi^\dagger_{n_4+1}(\boldsymbol{n})\} + \sum_{n_4,\boldsymbol{n}}\sum_k \{\chi^\dagger_{n_4}(\boldsymbol{n})\eta'_k(\boldsymbol{n})\chi_{n_4}(\boldsymbol{n}+\hat{\boldsymbol{k}}) + \chi^\dagger_{n_4}(\boldsymbol{n}+\hat{\boldsymbol{k}})\eta'_k(\boldsymbol{n})\chi_{n_4}(\boldsymbol{n})\}$$

$$\sum_{n_4}\sum_{\boldsymbol{n}}\{\chi^\dagger_{n_4}(\boldsymbol{n})\chi_{n_4+1}(\boldsymbol{n}) + \varphi^\dagger_{n_4}(\boldsymbol{n})\varphi_{n_4+1}(\boldsymbol{n})\}$$

$$\boldsymbol{I} = \int \prod_{n_4}\left[d\varphi^\dagger_{n_4} d\varphi_{n_4}\right] \prod_{\boldsymbol{n},n_4} \delta(\varphi_{n_4}(\boldsymbol{n}) - \chi^\dagger_{n_4}(\boldsymbol{n}))\delta(\varphi^\dagger_{n_4}(\boldsymbol{n}) - \chi_{n_4}(\boldsymbol{n})) \qquad \delta(\varphi^\dagger_{n_4}(\boldsymbol{n}) - \chi_{n_4}(\boldsymbol{n})) \equiv (\varphi^\dagger_{n_4}(\boldsymbol{n}) - \chi_{n_4}(\boldsymbol{n}))$$

$$\{\hat{\chi}(\boldsymbol{n})\,,\,\hat{\chi}^\dagger(\boldsymbol{n}')\} = \delta_{\boldsymbol{n},\boldsymbol{n}'}\,, \qquad \{\hat{\varphi}(\boldsymbol{n})\,,\,\hat{\varphi}^\dagger(\boldsymbol{n}')\} = \delta_{\boldsymbol{n},\boldsymbol{n}'}$$

$$\left|\chi_{n_4+1}, \varphi_{n_4+1}\right\rangle \equiv \exp\sum_{\boldsymbol{n}}\left\{\hat{\chi}^\dagger(\boldsymbol{n})\chi_{n_4+1}(\boldsymbol{n}) + \hat{\varphi}^\dagger(\boldsymbol{n})\varphi_{n_4+1}(\boldsymbol{n})\right\}|0\rangle\,,$$
$$\left\langle\chi^\dagger_{n_4}, \varphi^\dagger_{n_4}\right| \equiv \langle 0|\exp\sum_{\boldsymbol{n}}\left\{\chi^\dagger_{n_4}(\boldsymbol{n})\hat{\chi}(\boldsymbol{n}) + \varphi^\dagger_{n_4}(\boldsymbol{n})\hat{\varphi}(\boldsymbol{n})\right\}.$$

$$\boldsymbol{I} = \int\left[d\chi^\dagger_{n_4} d\chi_{n_4+1}\right]\left[d\varphi^\dagger_{n_4} d\varphi_{n_4+1}\right] \times \exp\left\{-\sum_{\boldsymbol{n}}[\chi^\dagger_{n_4}(\boldsymbol{n})\chi_{n_4+1}(\boldsymbol{n}) + \varphi^\dagger_{n_4}(\boldsymbol{n})\varphi_{n_4+1}(\boldsymbol{n})]\right\}\left|\chi_{n_4+1}, \varphi_{n_4+1}\right\rangle\left\langle\chi^\dagger_{n_4}, \varphi^\dagger_{n_4}\right|$$

$$\widehat{T}_F = \exp\left\{-\frac{1}{2}\sum_{\boldsymbol{n}}\sum_{k=1}^{3}\left[\hat{\chi}^\dagger(\boldsymbol{n}+\hat{\boldsymbol{k}})\eta'_k(\boldsymbol{n})\hat{\varphi}^\dagger(\boldsymbol{n}) + \hat{\chi}^\dagger(\boldsymbol{n})\eta'_k(\boldsymbol{n})\hat{\varphi}^\dagger(\boldsymbol{n}+\hat{\boldsymbol{k}})\right]\right\} \times \prod_{\boldsymbol{n}} :\left(\hat{\varphi}^\dagger(\boldsymbol{n}) - \hat{\chi}(\boldsymbol{n})\right)\left(\hat{\varphi}(\boldsymbol{n}) - \hat{\chi}^\dagger(\boldsymbol{n})\right): \times \exp\left\{-\frac{1}{2}\sum_{\boldsymbol{n}}\sum_{k=1}^{3}\left[\hat{\varphi}(\boldsymbol{n}+\hat{\boldsymbol{k}})\eta'_k(\boldsymbol{n})\hat{\chi}(\boldsymbol{n}) + \hat{\varphi}(\boldsymbol{n})\eta'_k(\boldsymbol{n})\hat{\chi}(\boldsymbol{n}+\hat{\boldsymbol{k}})\right]\right\}$$

$\widehat{T}_F$ is hermitian, but not positive definite. The Hamiltonian follows $(\widehat{T}_F)^2$

Chiral property of Wilson-Dirac fermions

- Axial Ward-Takahashi identity
- Lattice perturbation theory
    - Self-energy correction
    - Chiral anomaly
- Aoki Phase
- (Topology, Index theorem)

- Axial Ward-Takahashi identity

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi \, \mathrm{e}^{-(S_G[U] + S_W[\psi, \bar{\psi}, U])} \, \mathcal{O}$$

$$\psi(x) \to \psi(x) + i\gamma_5 \alpha(x) \psi(x)$$

$$\bar{\psi}(x) \to \bar{\psi}(x) + \bar{\psi}(x) i\gamma_5 \alpha(x)$$

$$\delta_\alpha S_{\mathrm{W}} = a^4 \sum_x \left\{ i\partial_\mu \alpha(x) J_\mu^A(x) + i\alpha(x) J^A(x) \right\}$$

$$J_\mu^A(x) = \frac{1}{2} \left[\psi(x) \gamma_\mu \gamma_5 \psi(x + \hat{\mu}) + \psi(x + \hat{\mu}) \gamma_\mu \gamma_5 \psi(x) \right]$$

$$J^A(x) = \frac{1}{2} \left[\psi(x) \cdot \nabla_\mu {\nabla_\mu}^\dagger \psi(x) + \psi \nabla_\mu {\nabla_\mu}^\dagger (x) \cdot \psi(x) \right] + 2m_0 \bar{\psi}(x) \gamma_5 \psi(x)$$

$$- \left\langle \left(\partial_\mu^* J_\mu^A(x) + J^A(x) \right) \mathcal{O} \right\rangle + \langle \delta_x \mathcal{O} \rangle = 0$$

- Lattice perturbation theory

[*Karsten, Smit (1981)*]
[*Kawai, Nakayama, Seo (1981)*]

$$Z[J_\mu,\eta,\bar\eta] = \int \mathcal{D}U\mathcal{D}\bar\psi\mathcal{D}\psi\, \mathrm{e}^{-(S_G[U]+S_W[\psi,\bar\psi,U])}\, \mathrm{e}^{(J_\mu\cdot A_\mu+\bar\eta\cdot\psi+\bar\psi\cdot\eta)}$$

$$U(x,\mu) = \sum_{l=0}^{\infty}\frac{(iag)^l}{l!}A_\mu(x)^l \qquad A_\mu(x) = \int\frac{d^4k}{(2\pi)^4}\mathrm{e}^{ik(x+\hat\mu/2)}A_\mu(k) \qquad \psi(x) = \int\frac{d^4p}{(2\pi)^4}\mathrm{e}^{ipx}\psi(p)$$

$$S_\mathrm{W} = \int\frac{d^4p}{(2\pi)^4}\bar\psi(p)D_{\mathrm{w}0}(p)\psi(p) + \sum_{l=1}^{\infty}V^{(l)}$$

$$D_{\mathrm{w}0}(p) = \gamma_\mu\frac{i}{a}\sin(p_\mu a) + \frac{r}{a}\sum_\mu[1-\cos(p_\mu a)] + m_0$$

$$V^{(l)} = \prod_{i=1}^{l}\int\frac{d^4k_l}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\bar\psi(q)\left[\frac{(iag)^l}{l!}(-i\partial_\mu)^l D_{\mathrm{w}0}(p+\sum_i k_i/2)\right]\psi(p)A_{\mu_1}(k_1)\cdots A_{\mu_l}(k_1)$$

$$av_\mu^{(1)}(k,p) = \gamma_\mu\cos(p+k/2)a - ri\sin(p+k/2)a$$

$$av_\mu^{(2)}(k_1,k_2,p) = \gamma_\mu i\sin(p+k_1/2+k_2/2)a - r\cos(p+k_1/2+k_2/2)a$$

- Lattice perturbation theory

[*Karsten, Smit (1981)*]
[*Kawai, Nakayama, Seo (1981)*]

$$S_0 = \frac{1}{g_0^2} \sum_{n,\mu,\nu} \mathrm{Tr}(1 - U_{n\eta\nu}) \qquad U_{n\mu\nu} \equiv U_{n\mu} U_{n+\hat{\mu},\nu} U^\dagger_{n+\hat{\nu},\mu} U^\dagger_{n\nu}$$

$$-\tfrac{1}{2} a^3 \sum_{n,\mu} \{\bar{\psi}_n (\lambda - \gamma_\mu) U_{n\mu} \psi_{n+\hat{\mu}} + \bar{\psi}_{n+\hat{\mu}} (\lambda + \gamma_\mu) U^\dagger_{n\mu} \psi_n - 2\lambda \bar{\psi}_n \psi_n\}$$

$$U_{n\mu} = \exp(i g_0 a A^a_{n\mu} T^a)$$

$$U(\phi + \delta\phi) = U(\phi)(1 + i\delta\phi^a E_{ab}(\phi) T^b) = (1 + i\delta\phi^a E_{ba}(\phi) T^b) U(\phi)$$

$$\delta A^a_{n\mu} = \frac{1}{g_0 a} \{E^{-1}_{ab}(g_0 a A_{n\mu}) \delta\omega^b_n - E^{-1}_{ba}(g_0 a A_{n\mu}) \delta\omega^b_{n+\hat{\mu}}\}$$

$$E_{ab}(\phi) = \left(\frac{e^{i\tilde{\phi}} - 1}{i\tilde{\phi}}\right)_{ab}$$

$$S_{\text{measure}} = -\tfrac{1}{2} \sum_{n,\mu} \mathrm{Tr} \ln \frac{2(1 - \cos g_0 a \tilde{A}_{n\mu})}{(g_0 a \tilde{A}_{n\mu})^2}$$

$$[\mathrm{d}U] = (\det g_{ab})^{1/2} \prod_a \mathrm{d}\phi^a$$

$$g_{ab}\, \mathrm{d}\phi^a\, \mathrm{d}\phi^b = \mathrm{Tr}\,(\mathrm{d}U\, \mathrm{d}U^\dagger)$$

$$S_{\text{gauge fixing}} = \frac{a^2}{2\alpha} \sum_n \left\{\sum_\mu (A^a_{n\mu} - A^a_{n-\hat{\mu},\mu})\right\}^2$$

$$\begin{aligned} g_{ab}\, \mathrm{d}\phi^a\, \mathrm{d}\phi^b &= \mathrm{Tr}\,\{U(\phi) i\, \mathrm{d}\phi^a E_{ac}(\phi) T^c (-i\, \mathrm{d}\phi^b) E_{bd}(\phi) T^d U^\dagger(\phi)\} \\ &= \tfrac{1}{2} E_{ac}(\phi) E^t_{cb}(\phi)\, \mathrm{d}\phi^a\, \mathrm{d}\phi^b \\ &= \left(\frac{1 - \cos\tilde{\phi}}{\tilde{\phi}^2}\right)_{ab} \mathrm{d}\phi^a\, \mathrm{d}\phi^b, \end{aligned}$$

$$\begin{aligned} S_{\text{FP}} = a^2 \sum_{n,\mu} \bar{c}^a_n [&\{E^{-1}_{ab}(g_0 a A_{n\mu}) c^b_n - E^{-1}_{ba}(g_0 a A_{n\mu}) c^b_{n+\hat{\mu}}\} \\ &- \{E^{-1}_{ab}(g_0 a A_{n-\hat{\mu},\mu}) c^b_{n-\hat{\mu}} - E^{-1}_{ba}(g_0 a A_{n-\hat{\mu},\mu}) c^b_n\}] \\ = -a^2 \sum_{n,\mu} (\bar{c}^a_{n+\hat{\mu}} - \bar{c}^a_n) &\{E^{-1}_{ab}(g_0 a A_{n\mu}) c^b_n - E^{-1}_{ba}(g_0 a A_{n\mu}) c^b_{n+\hat{\mu}}\} \end{aligned}$$

$$\frac{\delta_{\mu\nu}-(1-\alpha)\dfrac{\hat{k}_\mu\hat{k}_\nu}{\hat{k}^2}}{\hat{k}^2}\,\delta_{ab}$$

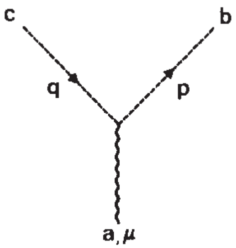
Gluon Propagator

$$\frac{1}{\hat{k}^2}\,\delta_{ab}$$

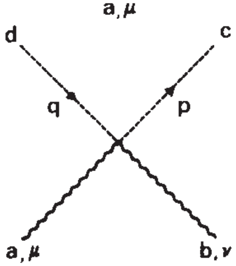
Ghost Propagator

$$\frac{1}{\frac{1}{a}\sum_\mu i\gamma_\mu \sin k_\mu a+\frac{1}{2}\lambda a\hat{k}^2}\,\delta_{ab}$$

Quark Propagator

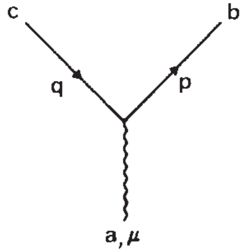


$$igf_{abc}\,\hat{p}_\mu\,\cos(\tfrac{1}{2}q_\mu a)$$

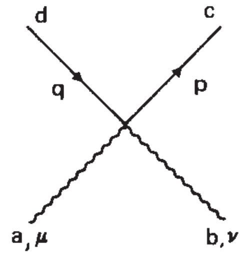


$$\frac{1}{12}\,g^2a^2(f_{ade}f_{bce}+f_{ace}f_{bde})\,\delta_{\mu\nu}\hat{p}_\mu\hat{q}_\mu$$

Ghost – Gluon Vertex

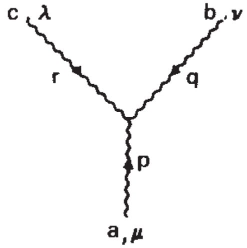


$$-ig\left\{\gamma_\mu\cos\tfrac{1}{2}(p+q)_\mu a-\tfrac{1}{2}i\lambda a\widehat{(p+q)_\mu}\right\}(T^a)_{bc}$$



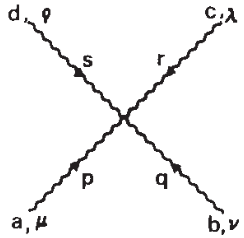
$$-\tfrac{1}{2}g^2a\delta_{\mu\nu}\left\{\lambda\cos\tfrac{1}{2}(p+q)_\mu a-\tfrac{1}{2}ia\gamma_\mu\widehat{(p+q)_\mu}\right\}\{T^a,T^b\}_{cd}$$

Quark – Gluon Vertex



$$igf_{abc}\left\{\delta_{\nu\lambda}\widehat{(r-q)}_\mu\cos\tfrac{1}{2}p_\nu a+\delta_{\lambda\mu}\widehat{(p-r)}_\nu\cos\tfrac{1}{2}q_\lambda a+\delta_{\mu\nu}\widehat{(q-p)}_\lambda\cos\tfrac{1}{2}r_\mu a\right\}$$

3–Gluon Vertex



see the text

4–Gluon Vertex

$$- \left\langle \left(\partial^*_\mu J^A_\mu(x) + J^A(x) \right) \mathcal{O} \right\rangle + \langle \delta_x \mathcal{O} \rangle = 0$$

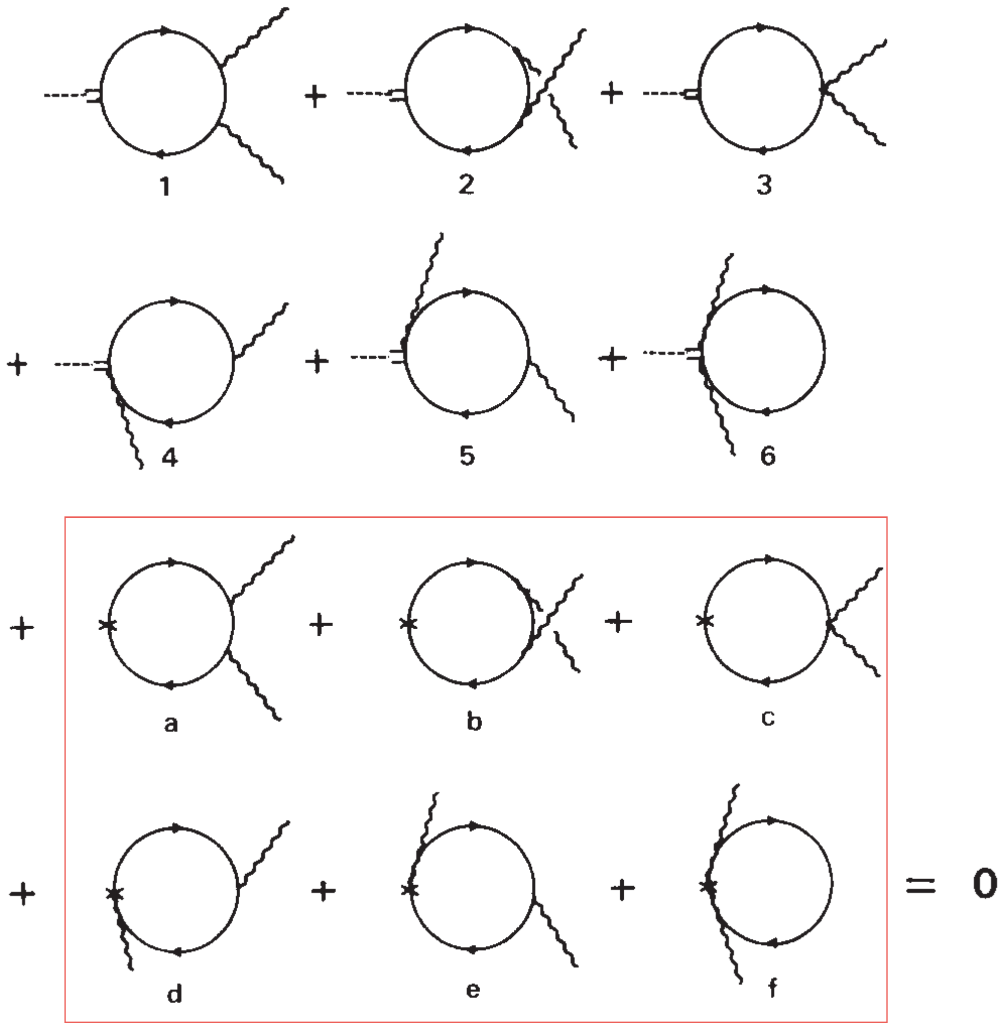
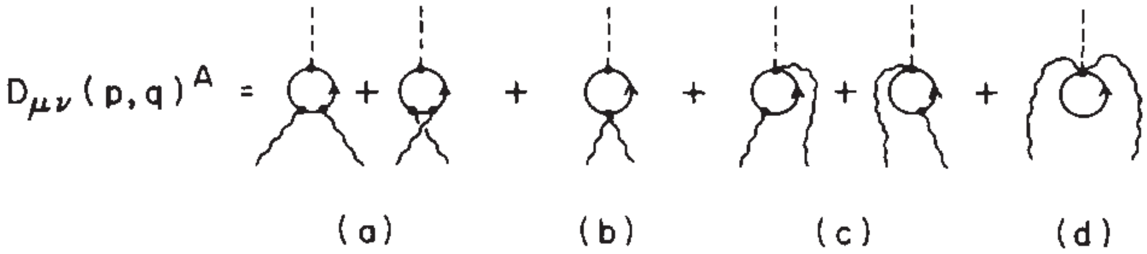


Fig. 6. A Ward identity for the axial vector current.

- Chiral anomaly

$$\left\langle J^A(x) A_\mu(y_1) A_\mu(y_2) \right\rangle$$



$$D^{A}_{\mu\nu}(p,q) = -2g^2 \int_l \mathrm{Tr}\, \{ i\gamma_5 a^{-1} [\mathcal{M}(al+aq) + \mathcal{M}(al-ap)]$$

$$\times S(l-p) V_\mu(l-p,l) S(l) V_\nu(l,l+q) S(l+q) \}$$

$$D^{A}_{\mu\nu}(p,q)|_{M=M_c} = ig^2 \varepsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta 16 \int_l \cos l_\mu \cos l_\nu \cos l_\alpha$$

$$\times \frac{[\mathcal{M}_c^2(l) \cos l_\beta - 4r\mathcal{M}_c(l) \sin^2 l_\beta]}{[\mathcal{M}_c^2(l) + \sin^2 l]^3},$$

$$\mathcal{M}_c(ak) = r \sum_\lambda (1 - \cos ak_\lambda)$$

$$\mathcal{M}_c^2 \cos l_\beta - 4r\mathcal{M}_c \sin^2 l_\beta = \cos l_\beta (\mathcal{M}_c^2 + 4\sin^2 l_\beta)$$

$$+ (\mathcal{M}_c^2 + \sin^2 l)^3 \sin l_\beta \frac{\partial}{\partial l_\beta} (\mathcal{M}_c^2 + \sin^2 l)^{-2}$$

$$-iD^{A}_{\mu\nu}(p,q)|_{M=M_c} = -\frac{g^2}{2\pi^2} \varepsilon_{\mu\nu\alpha\beta} p_\alpha q_\beta$$

$$\varepsilon_{\mu\nu\lambda\rho} p_\lambda q_\rho c(\lambda)$$

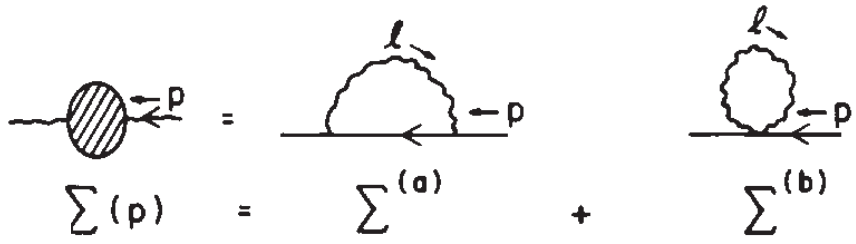
$$c(\lambda) = 16g^2 \int \frac{d^4k}{(2\pi)^4} \left[\frac{\cos k_\mu \cos k_\nu \cos k_\lambda \cos k_\rho (4 \sin^2 k_\rho - \sum_\sigma \sin^2 k_\sigma)}{\{\sum_\sigma \sin^2 k_\sigma + 4\lambda^2 (\sum_\sigma \sin^2 [\frac{1}{2} k_\sigma])^2\}^3} \right.$$

$$\left. + \frac{\partial}{\partial k_\rho} \frac{\cos k_\mu \cos k_\nu \cos k_\lambda \sin k_\rho}{\{\sum_\sigma \sin^2 k_\sigma + 4\lambda^2 (\sum_\sigma \sin^2 [\frac{1}{2} k_\sigma])^2\}^2} \right],$$

$$c(\lambda) = \lim_{D \to 4} 16g^2 \int \frac{d^D k}{(2\pi)^D} \frac{\cos k_\mu \cos k_\nu \cos k_\lambda \cos k_\rho \sum_{\sigma=5}^{D} \sin^2 k_\sigma}{\{\sum_\sigma \sin^2 k_\sigma + 4\lambda^2 (\sum_\sigma \sin^2 [\frac{1}{2} k_\sigma])^2\}^3}$$

$$= -\frac{g^2}{2\pi^2}.$$

- Self-energy correction



$$\Sigma(p) = \Sigma^{(a)}(p) + \Sigma^{(b)}(p) ,$$

$$\Sigma^{(a)}(p) = -\bar{g}^2 \int_l V_\mu(p, l+p) S(l+p) V_\nu(l+p, p) D_{\mu\nu}(l)$$

$$\Sigma^{(b)}(p) = \bar{g}^2 \int_l V_{\mu\nu}(p, p) D_{\mu\nu}(l) ,$$

$$\bar{g}^2 = \tfrac{1}{2} g^2 (N^2 - 1)/N$$

$$\mathcal{M}(k) = aM - r \sum_\nu \cos k_\nu = am + r \sum_\nu (1 - \cos k_\nu) - a\delta m$$

$$S'(p)^{-1} = M - 4ra + \not{p} + \Sigma(p)$$

$$aM_c = 4r - \bar{g}^2 \sigma_1(r) + O(g^4)$$

$$\sigma_1(r) = r \int_l \frac{\sin^2 l + 2 \sin^2 \frac{1}{2}l [\cos^2 l + (1 - r^2) \sin^2 \frac{1}{2}l]}{[(2r \sin^2 \frac{1}{2}l)^2 + \sin^2 l] 4 \sin^2 \frac{1}{2}l}$$

$$\Sigma(p) = \Sigma_0(p^2) + \not{p}\Sigma_1(p^2)$$

$$\tilde{S}_0(p^2) = Z_2[m - \delta m + \Sigma_0(p^2)]$$

$$\tilde{S}_1(p^2) = Z_2[1 + \Sigma_1(p^2)]$$

$$Z_2 = 1 + \bar{g}^2\left[\frac{(1-\zeta)}{16\pi^2}\ln a^2\mu^2 - \sigma_3(r) + C\right],$$

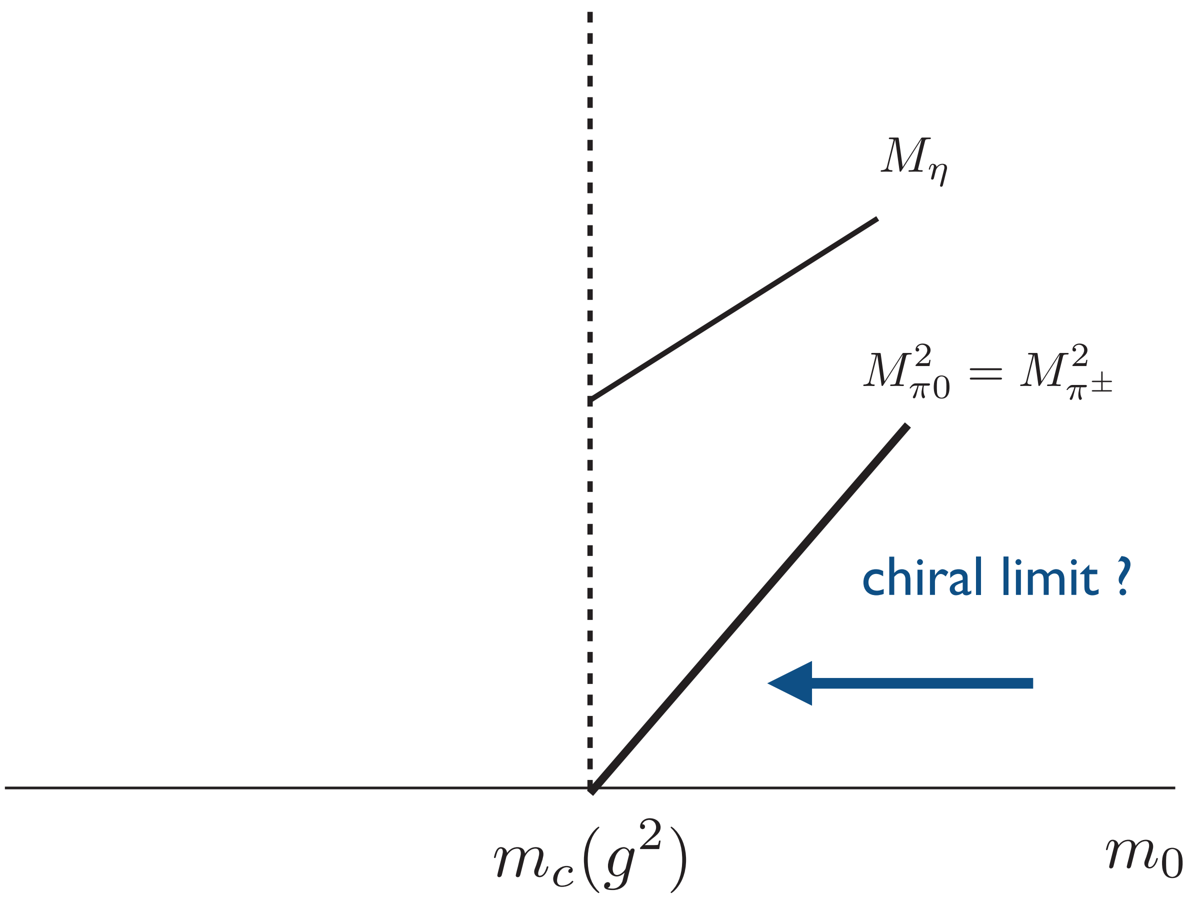
$$\delta m = \bar{g}^2\sigma_1(r)a^{-1} + m(1 - Z_m),$$

$$Z_m = 1 - (Z_2 - 1) + \bar{g}^2\left[\frac{\ln a^2\mu^2}{4\pi^2} - \sigma_2(r) + C'\right] + \mathrm{O}(g^4)$$

- Aoki Phase

N_f=2 Lattice QCD with Wilson fermions

No exact chiral symmetry chiral limit ?



- Aoki Phase

[S.Aoki (1984)]

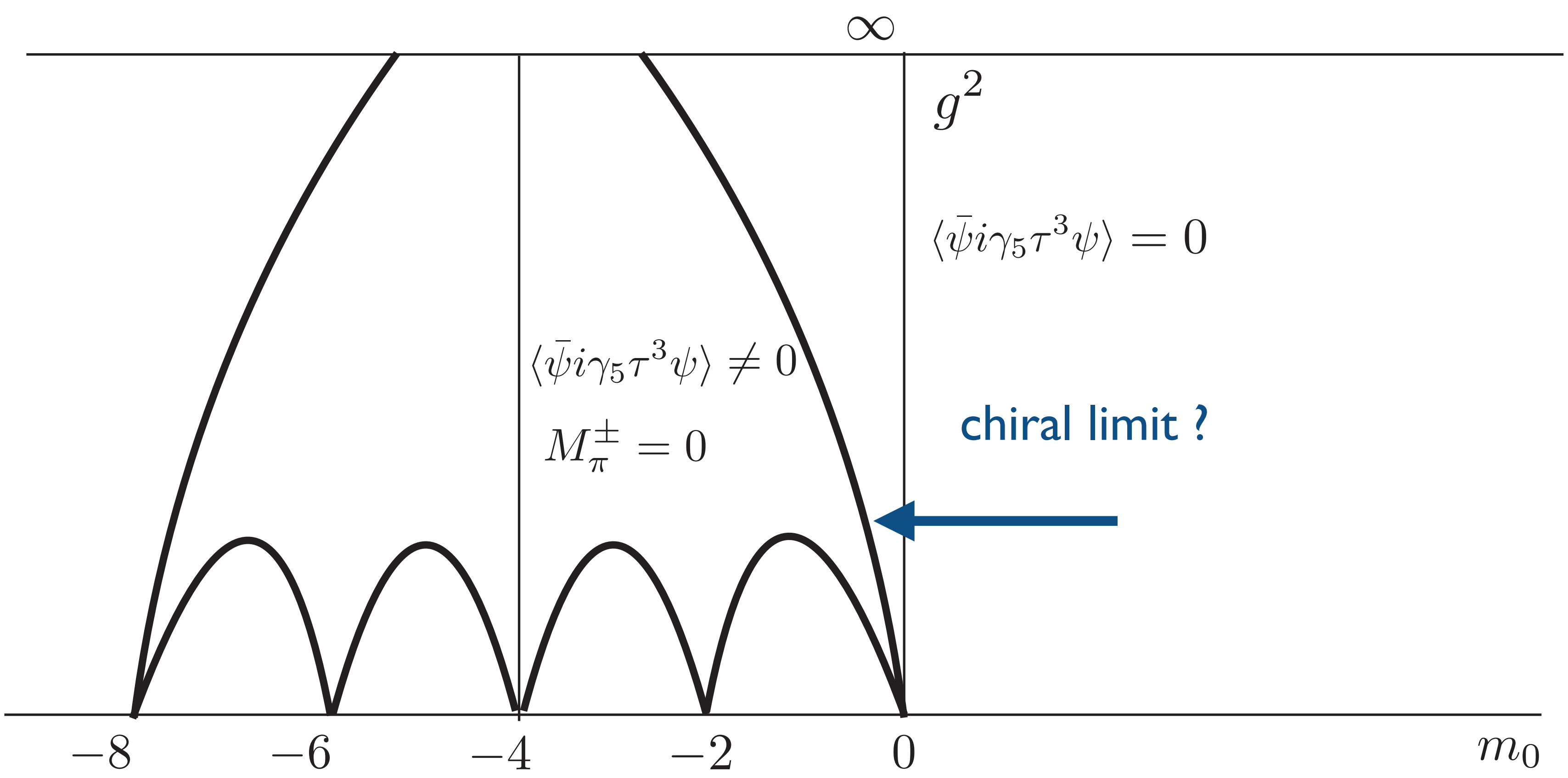
N_f=2 Lattice QCD with Wilson fermions

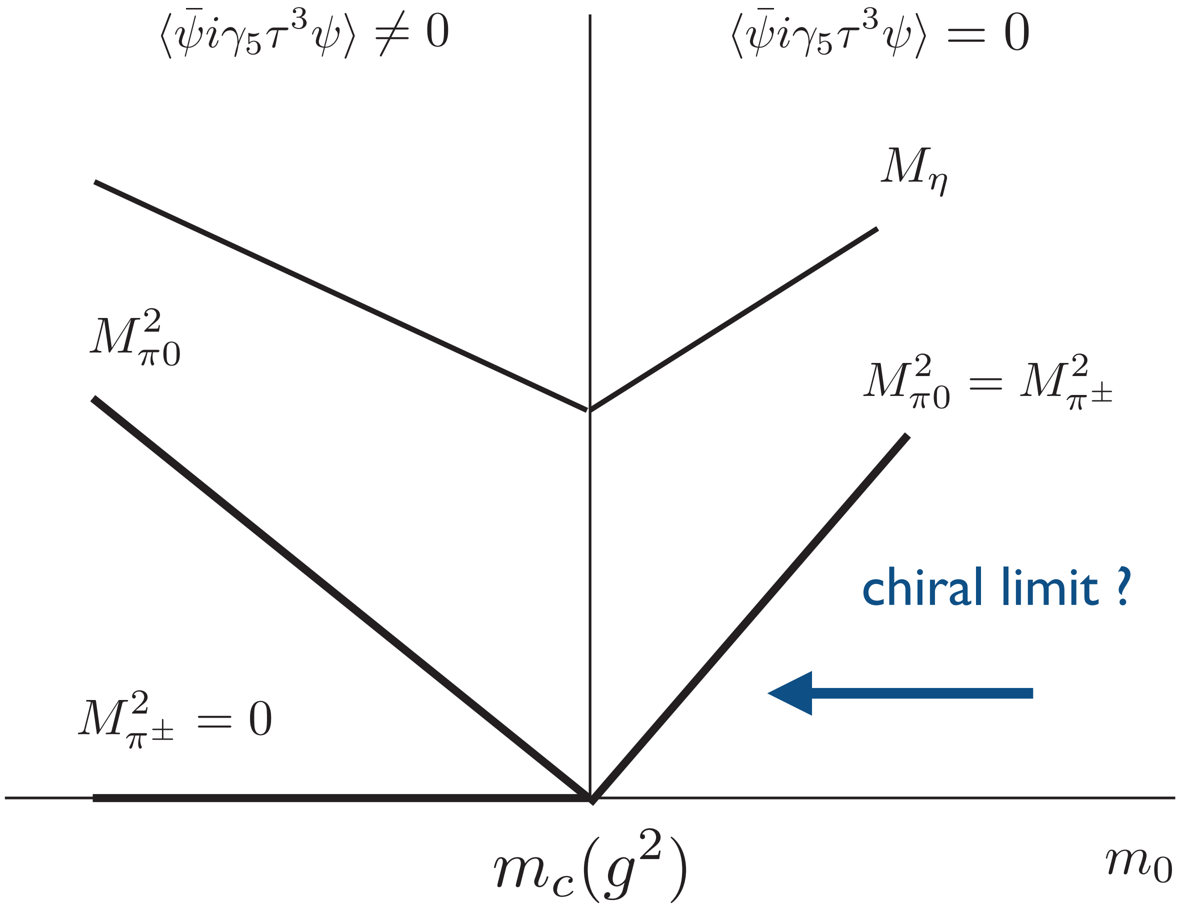
No exact chiral symmetry

Exact Parity, Flavor symmetries

M_π —> 0 as m_0 —> $m_c(g)$

SSB $\langle \bar{\psi} i\gamma_5 \tau^3 \psi \rangle \neq 0$





$\langle \bar{\psi} i \gamma_5 \tau^3 \psi \rangle \neq 0$
$\langle \bar{\psi} i \gamma_5 \tau^3 \psi \rangle = 0$
M_η
$M^2_{\pi 0}$
$M^2_{\pi 0} = M^2_{\pi \pm}$
chiral limit ?
$M^2_{\pi \pm} = 0$
$m_c(g^2)$
m_0

Ginsparg-Wilson relation

- Ginsparg-Wilson relation defines the chiral limit of lattice fermion action

$$\gamma_5 D_*^{-1} + D_*^{-1}\gamma_5 = 2a\gamma_5\delta_{xy}$$

  - Low energy effective lattice action at IR fixed point of block spin tr.

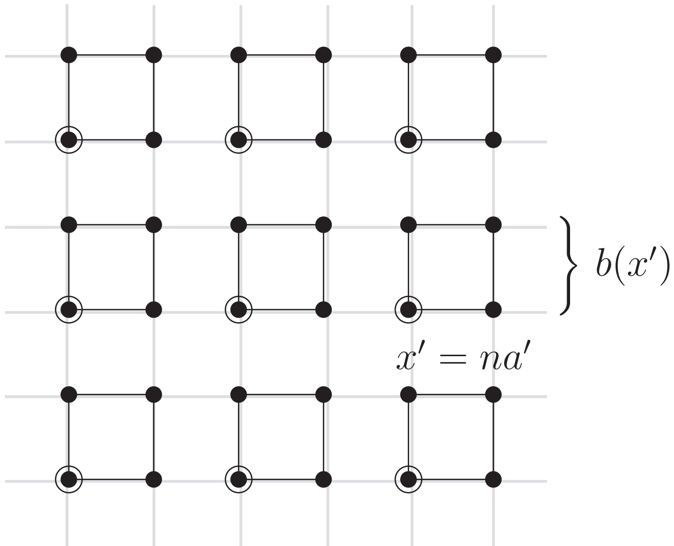
[*Ginsparg-Wilson(1982)*]

  - Exact chiral symmetry emerges [*Luscher (1999)*]

$$\delta S = 0 \qquad \delta_\alpha\psi(x) = i\alpha\,\gamma_5(1-2aD)\psi(x), \quad \delta_\alpha\bar{\psi}(x) = i\alpha\,\bar{\psi}(x)\gamma_5$$

Block-spin transformation

Ginsparg-Wilson(1982)



$$\psi'(x') \Leftarrow \frac{Z}{2^4} \sum_{x \in b(x')} \psi(x)$$

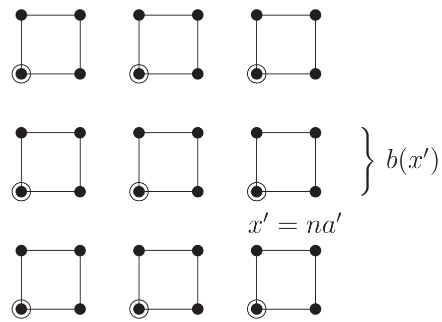
$$e^{-S'[\psi',\bar{\psi}']} = \int \prod_x d\psi(x) d\bar{\psi}(x)\; e^{-S_W[\psi,\bar{\psi}]} \times$$
$$\exp\left\{ -\alpha \sum_{x'} \left(\bar{\psi}'(x') - \bar{\Psi}(x';\bar{\psi}) \right) \left(\psi'(x') - \Psi(x';\psi) \right) \right\}$$

IR fixed point :

$$S^* = a^4 \sum_x \bar{\psi}(x) D^* \psi(x)$$ local, low-energy effective action

$$\gamma_5 D_*^{-1} + D_*^{-1} \gamma_5 = \frac{2}{\alpha} a \gamma_5 \delta_{xy}$$ GW rel.

Block spin transformation for Wilson-Dirac fermions



$$B_{x'} = \{x \in \mathbb{Z}^d \,\big|\, x = x' + \hat{\mu}, \mu = 1, \cdots, d\},\ x' \in (2\mathbb{Z})^d$$

$$\mathrm{e}^{-S'[\psi'(x'),\bar{\psi}'(x')]} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\,\mathcal{D}\eta'\mathcal{D}\bar{\eta}'\,\mathrm{e}^{-S[\psi(x),\bar{\psi}(x)]} \times$$
$$\mathrm{e}^{\sum_{x'}\left\{a\,\bar{\eta}'(x')\eta'(x')+\bar{\eta}'(x')\left(\psi'(x')-\frac{b}{2^d}\sum_{x\in B_{x'}}\psi(x)\right)+\eta'(x')\left(\bar{\psi}'(x')-\frac{b}{2^d}\sum_{x\in B_{x'}}\bar{\psi}(x)\right)\right\}}$$

$$\mathrm{e}^{W[\eta(x),\bar{\eta}(x)]}\,Z = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ \mathrm{e}^{-S[\psi(x),\bar{\psi}(x)]}\,\mathrm{e}^{-\sum\left\{\bar{\eta}(x)\psi(x)+\eta(x)\bar{\psi}(x)\right\}}$$

$$\mathrm{e}^{W'[\eta'(x'),\bar{\eta}'(x')]} = \mathrm{e}^{W[\eta(x),\bar{\eta}(x)]}\Big|_{\eta(x),\bar{\eta}(x)\to\frac{b}{2^d}\eta'(x'),\,\frac{b}{2^d}\bar{\eta}'(x'),\,x\in B_{x'}} \times \mathrm{e}^{a\sum_{x'}\bar{\eta}'(x')\eta'(x')}$$

$$\mathrm{e}^{W[\eta(x),\bar{\eta}(x)]} = \mathrm{e}^{-\sum_{x,y}\bar{\eta}(x)D_W{}^{-1}(x,y)\eta(y)}$$

Fixed point theory *D* (Dirac fermion)*

$$\mathrm{e}^{W_{D^*}[\eta(x),\bar{\eta}(x)]} = \mathrm{e}^{-\sum_{x,y}\bar{\eta}(x){D_*}^{-1}(x,y)\eta(y)}$$

$$b = 2^{\frac{d-1}{2}}$$

$$\begin{aligned}
{D_*}^{-1}(x,y) &= \int \frac{d^dk}{(2\pi)^4}\,\mathrm{e}^{ik(x-y)}\left\{-i\gamma_\mu\alpha_\mu(k) + R\right\}, \\
\alpha_\mu(k) &= \sum_{l\in\mathbb{Z}^d}\prod_\nu\left(\frac{2\sin(k_\nu/2)}{k_\nu+2\pi l_\nu}\right)^2\frac{k_\mu+2\pi l_\mu}{(k_\mu+2\pi l_\mu)^2}, \\
R &= \frac{a}{\left(1-\frac{b^2}{2^d}\right)} = 2a.
\end{aligned}$$

$${D_c}^{-1}(x,y) = \int \frac{d^dk}{(2\pi)^4}\,\mathrm{e}^{ik(x-y)}\left\{-i\gamma_\mu\alpha_\mu(k)\right\}$$

$$D_*\gamma_5 + \gamma_5 D_* = 2R\,D_*\gamma_5 D_*$$

$$\begin{aligned}
S_{D^*} &= \sum_x \bar{\psi}(x)D_*\psi(x) \\
&= \sum_x\left\{\bar{\psi}_+(x)D_*\psi_+(x) + \bar{\psi}_-(x)D_*\psi_-(x)\right\}
\end{aligned}$$

Overlap Dirac operator

- overalp Dirac operator is a local, gauge-covariant solution to G-W rel.

[*Neuberger (1998)*]

$$D = \frac{1}{2a}\left(1 + X\frac{1}{\sqrt{X^\dagger X}}\right), \quad X = aD_{\rm w} - m_0, \quad X^\dagger = \gamma_5 X \gamma_5$$

$$D_{\rm w} = -\gamma_\mu \frac{1}{2}(\nabla_\mu - \nabla_\mu^\dagger) + \frac{a}{2}\nabla_\mu \nabla_\mu^\dagger$$

  - Low energy effective lattice action of 4+1 dim. DW fermion
  - Index theorem holds true on the lattice (at a finite lattice spacing "*a*")
  - Reflection positivity is satisfied (in the free fermion limit at least)

Locality of overlap Dirac operator

$$D = \frac{1}{2a}\left(1 + \gamma_5 \frac{H_{\rm w}}{\sqrt{H_{\rm w}^2}}\right)$$

- Locality

1. If $0 < \alpha < H_w^2 < \beta$

   $\frac{1}{\sqrt{H_w^2}} = \frac{\kappa}{\sqrt{1-2tz+t^2}} = \kappa \sum_{k=0}^{\infty} t^k P_k(z)$

   $z = \frac{\beta+\alpha-2H_w^2}{\beta-\alpha}$, $\cosh^{-1}\theta = \frac{\beta-\alpha}{\beta+\alpha}$, $t = e^{-\theta}$, $\kappa = \sqrt{\frac{4t}{\beta-\alpha}}$

$$\left\| \frac{1}{\sqrt{H_w^2}}(x,y) \right\| < \frac{\kappa}{1-t} \exp\{-\theta|x-y|/2\}$$

2. If $\| 1 - U_{\Box} \| \leq \epsilon$, $\epsilon < \frac{1}{30}$ (admissibility condition)

$$H_w^2 = (aD_w - 1)(aD_w - 1)^\dagger \geq 1 - 30\epsilon > 0$$

cf. in the continuum limit

$$(a\gamma_\mu D_\mu - 1)^\dagger (a\gamma_\mu D_\mu - 1) = 1 - a^4 (D_\mu)^2 - ia^4 \frac{[\gamma_\mu, \gamma_\nu]}{4} F_{\mu\nu}$$

$$
\begin{aligned}
a\nabla_\mu\phi(x) &= U(x,\mu)\phi(x+\hat{\mu})-\phi(x),\\
a^2\nabla_\nu\nabla_\mu\phi(x) &= U(x,\nu)\left(U(x+\hat{\nu},\mu)\phi(x+\hat{\mu}+\hat{\nu})-\phi(x+\hat{\nu})\right)-\left(U(x,\mu)\phi(x+\hat{\mu})-\phi(x)\right)\\
a^2\left[\nabla_\mu,\nabla_\nu\right]\phi(x) &= \left\{U(x,\mu)U(x+\hat{\mu},\nu)-U(x,\nu)U(x+\hat{\nu},\mu)\right\}\phi(x+\hat{\mu}+\hat{\nu})\\
&= -\left\{1-U(x,\mu)U(x+\hat{\mu},\nu)U(x+\hat{\nu},\mu)^{-1}U(x,\nu)^{-1}\right\}\times\\
&\qquad U(x,\nu)U(x+\hat{\nu},\mu)\phi(x+\hat{\mu}+\hat{\nu})
\end{aligned}
$$

Therefore

$$
\parallel 1-U(x,\mu)U(x+\hat{\mu},\nu)U(x+\hat{\nu},\mu)^{-1}U(x,\nu)^{-1}\parallel\leq\epsilon\quad\Rightarrow\quad\parallel a^2\left[\nabla_\mu,\nabla_\nu\right]\parallel\leq\epsilon
$$

For $m_0 = 1$, $H_w^2 = (aD_w - 1)^\dagger (aD_w - 1)$ is evaluated as follows:

$$(aD_w - 1)^\dagger (aD_w - 1) = 1 + \frac{1}{4} \sum_{\mu \neq \nu} \{B_{\mu\nu} + C_{\mu\nu} + D_{\mu\nu}\}$$

$$\begin{aligned}
B_{\mu\nu} &= a^4 \nabla_\mu^* \nabla_\mu \nabla_\nu^* \nabla_\nu = a^4 \nabla_\mu^* \nabla_\nu^* \nabla_\nu \nabla_\mu - a^3 \nabla_\mu^* \left[\nabla_\mu, \nabla_\nu^* - \nabla_\nu\right] \\
C_{\mu\nu} &= \frac{1}{2} i\sigma_{\mu\nu} a^2 \left[\nabla_\mu^* + \nabla_\mu, \nabla_\nu^* + \nabla_\nu\right] \\
D_{\mu\nu} &= -\gamma_\mu a^2 \left[\nabla_\mu^* + \nabla_\mu, \nabla_\nu^* - \nabla_\nu\right]
\end{aligned}$$

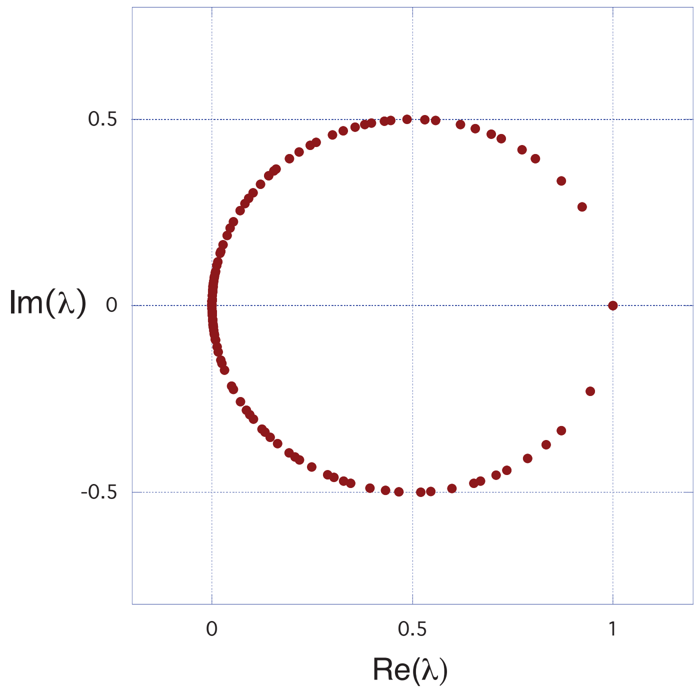
$$\begin{aligned}
a^2 H_w^2 &\geq 1 - \frac{1}{4} \sum_{\mu \neq \nu} \| -a^3 \nabla_\mu^* \left[\nabla_\mu, \nabla_\nu^* - \nabla_\nu\right] \| \\
&\quad - \frac{1}{4} \sum_{\mu \neq \nu} \| \frac{1}{2} i\sigma_{\mu\nu} a^2 \left[\nabla_\mu^* + \nabla_\mu, \nabla_\nu^* + \nabla_\nu\right] \| - \frac{1}{4} \sum_{\mu \neq \nu} \| -\gamma_\mu a^2 \left[\nabla_\mu^* + \nabla_\mu, \nabla_\nu^* - \nabla_\nu\right] \| \\
&= 1 - \frac{1}{4} \cdot 12 \cdot 2 \cdot 2\epsilon - \frac{1}{4} \cdot \frac{1}{2} \cdot 12 \cdot 4\epsilon - \frac{1}{4} \cdot 12 \cdot 4\epsilon = 1 - 30\epsilon
\end{aligned}$$

cf. (in the continuum limit)

$$\begin{aligned}
(a\gamma_\mu D_\mu - 1)^\dagger (a\gamma_\mu D_\mu - 1) &= 1 - a^4 (D_\mu)^2 - a^4 \frac{[\gamma_\mu, \gamma_\nu]}{4} [D_\mu, D_\nu] \\
&= 1 - a^4 (D_\mu)^2 - i a^4 \frac{[\gamma_\mu, \gamma_\nu]}{4} F_{\mu\nu}
\end{aligned}$$

Index theorem on the lattice

Eigenvalue distribution :



Zero modes : $D\psi_0 = 0$

$$\gamma_5 \psi_0 = \pm \psi_0$$

$$\because \quad D\gamma_5\psi_0 = (-\gamma_5 D + 2aD\gamma_5 D)\psi_0 = 0$$

$$\mathrm{Index}(D) = n_+ - n_-$$

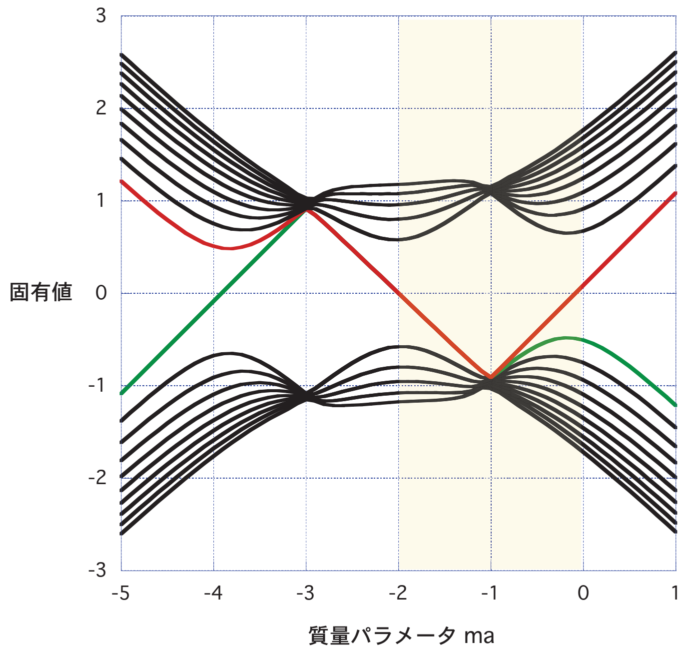
Topological charge = spectral flow of H_w

= chiral anomaly

(Jacobian of the chiral tr.)

$$Q = -\frac{1}{2}\mathrm{Tr}\left\{\frac{H_\mathrm{w}}{\sqrt{H_\mathrm{w}^2}}\right\} = \mathrm{Tr}\gamma_5(1 - aD)$$

$$(-2 < -m_0 a < 0)$$



Index theorem :

$$\mathrm{Index}(D) = \mathrm{Tr}\gamma_5(1 - aD)\,\mathbf{(x,x)}$$

Overlap Dirac operator is normal and satisfies the γ_5-conjugate relation:

$$D + D^\dagger = 2aD^\dagger D = 2aDD^\dagger \text{ (normal)}, \quad D^\dagger = \gamma_5 D \gamma_5 \text{ (γ_5-conjugate)}$$

1. Show that the eivenvalues of D,

$$D\psi_\lambda = \lambda\psi_\lambda,$$

distribute on the circle with the radius $1/2a$ and centered at $(1/2a, 0)$ in the two-dimensional complex plane.

2. Show that the eivenvalues of D are classified into three groups as follows:

$$\begin{array}{rll} \lambda = 0: & \gamma_5\psi_\lambda(x) = \pm\psi_\lambda(x) & n_\pm \\ \lambda = 1/a: & \gamma_5\psi_\lambda(x) = \pm\psi_\lambda(x) & N_\pm \\ \lambda \neq 0, 1/a: & \text{pair-wise} \left\{ \begin{array}{l} \lambda \to \psi_\lambda \\ \lambda^* \to \gamma_5\psi_\lambda \end{array} \right. & \end{array}$$

3. Prove the index theorem on the lattice

$$\mathrm{Tr}\{\gamma_5(1 - aD)\} = n_+ - n_-$$

---

1. For an eigenvalue λ and an eivenvector $\psi_\lambda(x)$ belonging to it,

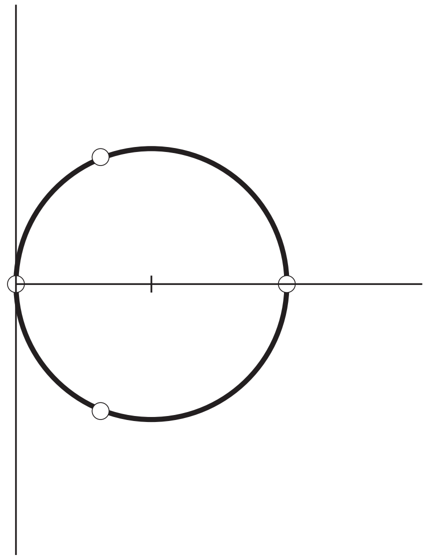
$$D\,\psi_\lambda(x) = \lambda\,\psi_\lambda(x),$$

one has

$$a^4 \sum_x \psi_\lambda^\dagger(x) \left\{ D + D^\dagger - 2aD^\dagger D \right\} \psi_\lambda(x) = (\lambda + \lambda^* - 2a\lambda^*\lambda)\,(\psi_\lambda, \psi_\lambda) = 0$$

Then one can show

$$\lambda + \lambda^* - 2a\lambda^*\lambda = (-2a)\left[(\lambda - 1/2a)(\lambda - 1/2a)^* - (1/2a)^2\right] = 0$$



2. (a) $\lambda = 0$

Suppose ψ_λ belongs to the zero eigenvalue, $D\psi_\lambda = 0$. Then

$$D\left(\gamma_5\psi_\lambda\right) = \{-\gamma_5 D + 2aD\gamma_5 D\}\,\psi_\lambda = 0.$$

This implies that ψ_λ can be made to be chiral

$$D\left(\frac{1+\gamma_5}{2}\right)\psi_\lambda = 0 \quad \text{or} \quad D\left(\frac{1-\gamma_5}{2}\right)\psi_\lambda = 0$$

(b) $\lambda = 1/a$

Suppose ψ_λ belongs to the eigenvalue $1/a$, $D\psi_\lambda = (1/a)\psi_\lambda$. Then

$$(D - 1/a)\left(\gamma_5\psi_\lambda\right) = \{-\gamma_5(D-1/a) + 2a(D-1/a)\gamma_5(D-1/a)\}\,\psi_\lambda = 0.$$

This implies that ψ_λ can be made to be chiral

$$(D-1/a)\left(\frac{1+\gamma_5}{2}\right)\psi_\lambda = 0 \quad \text{or} \quad (D-1/a)\left(\frac{1-\gamma_5}{2}\right)\psi_\lambda = 0$$

(c) $\lambda \neq 0, 1/a$

Suppose ψ_λ belongs to the eigenvalue $\lambda \neq 0, 1/a$, $D\psi_\lambda = \lambda\psi_\lambda$. Since D is normal, $\psi_\lambda^\dagger D = \lambda\psi_\lambda^\dagger$. Then

$$D\left(\gamma_5\psi_\lambda\right) = \gamma_5 D^\dagger\psi_\lambda = \gamma_5\left(\lambda\psi_\lambda^\dagger\right)^\dagger = \lambda^*\left(\gamma_5\psi_\lambda\right)$$

3.

$$
\begin{aligned}
\sum_{\lambda} \psi_\lambda^\dagger \gamma_5 (1 - aD) \psi_\lambda &= \sum_{\lambda=0,1/a} \psi_\lambda^\dagger \gamma_5 \psi_\lambda - a \sum_{\lambda=1/a} \lambda\, \psi_\lambda^\dagger \gamma_5 \psi_\lambda \\
&= (n_+ - n_-) + (N_+ - N_-) - (N_+ - N_-) \\
&= n_+ - n_-
\end{aligned}
$$

Locality, Topology of lattice gauge fields, Admissibility condition

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

$$\delta_\alpha \psi(x) = i\alpha\, \gamma_5 (1 - 2aD)\psi(x), \quad \delta_\alpha \bar{\psi}(x) = i\alpha\, \bar{\psi}(x)\gamma_5$$

Locality of D $\Longrightarrow$ GW rel. $\gamma_5 D + D\gamma_5 = 2D\gamma_5 D$ $\Longrightarrow$ Index of D

$\Downarrow$ Index theorem on the lattice

GW rel. $\searrow$ Local chiral anomaly $\sum_x -\mathrm{tr}\gamma_5 D(x,x)$ $\searrow$

$\Uparrow$ (Admissibility cond. $\Rightarrow$ Locality of D)

Admissibility cond. $\|1 - U_\Box(x)\| < \epsilon$ $\Longrightarrow$ Topological structure of lattice gauge fields $\Longrightarrow$ Topological charge $Q = -\frac{1}{2}\sum_x \mathrm{tr}\gamma_5 D$ $\Uparrow$ (Index theorem on the lattice)

$\Longrightarrow$ Mass gap of 4+1 dim. Domain-wall fermions (w/o bdy)

Weyl Fermion on the lattice (cf. NN theorem)

- Overlap Weyl fermions

$$\hat{\gamma}_5 = \gamma_5(1-2aD) \qquad \hat{\gamma}_5{}^2 = \mathbb{I} \qquad \hat{P}_\pm = \left(\frac{1\pm\hat{\gamma}_5}{2}\right), \quad P_\pm = \left(\frac{1\pm\gamma_5}{2}\right)$$

$$\hat{\gamma}_5 \psi_\pm(x) = \pm\,\psi_\pm(x), \qquad \bar{\psi}_\pm(x)\gamma_5 = \mp\,\bar{\psi}_\pm(x)$$

$$\begin{aligned} S &= a^4 \sum_x \bar{\psi}(x)\, D\, \psi(x) \\ &= a^4 \sum_x \left\{ \bar{\psi}(x) P_+ D \hat{P}_- \psi(x) + \bar{\psi}(x) P_- D \hat{P}_+ \psi(x) \right\} \end{aligned}$$

- Path integral measure and Chiral determinant

$$\psi_-(x) = \sum_i v_i(x) c_i \qquad \bar{\psi}_-(x) = \sum_i \bar{c}_i \bar{v}_i(x)$$

$$\{v_i(x) \mid \hat{\gamma}_5 v_i(x) = -v_i(x) \ (i=1,\cdots,N_-)\}$$
$$\{\bar{v}_i(x) \mid \bar{v}_i(x)\gamma_5 = +\bar{v}_i(x) \ (i=1,\cdots,\bar{N}_-)\}$$

$$Z = \int \prod_i dc_i \prod_j d\bar{c}_j \, \mathrm{e}^{-\sum_{ij} \bar{c}_j M_{ji} c_i} = \det M_{ji}$$

$$M_{ji} = a^4 \sum_x \bar{v}_j D v_i(x)$$

$(N_- \times \bar{N}_- \ \text{ rectangular matrix})$

[*Narayanan-Neuberger (1993)*]

- Gauge anomaly of overlap Weyl fermion

$$\mathfrak{U} = \left\{ \{U(x,\mu)\} \,\middle|\, \| 1 - U_{\mu\nu}(x) \| < \epsilon \ {}^{\forall}(x,\mu,\nu) \right\}$$

$$\mathcal{O} \subset \mathfrak{U}[Q] \qquad \tilde{v}_i(x) = v_l(x) \left(\mathcal{Q}^{-1}\right)_{li}, \quad \tilde{c}_j = \sum_l \mathcal{Q}_{jl} c_l$$

$$\mathcal{D}[\psi_-] \to \mathcal{D}[\psi_-] \det \mathcal{Q} \qquad \det M_{ji} \to \det M_{ji} \det \mathcal{Q}$$

$$\delta_\eta U(x,\mu) = i a \eta_\mu(x) U(x,\mu)$$

$$\delta_\eta \ln \det M_{ji} = \mathrm{Tr}\{\delta_\eta D \hat{P}_- D^{-1} P_+\} + \sum_i (v_i, \delta_\eta v_i)$$

$$= i \mathrm{Tr} \omega \gamma_5 (1 - D) - i \sum_i (v_i, \delta_\omega v_i) \qquad \eta_\mu(x) = -i \nabla_\mu \omega(x)$$

$$\mathrm{tr}\{T^a \gamma_5 (1 - aD)(x,x)\} \xrightarrow{a \to 0} \frac{-1}{128\pi^2} d^{abc} \epsilon_{\mu\nu\rho\sigma} F^b_{\mu\nu}(x) F^c_{\rho\sigma}(x) + O(a)$$ **gauge anomaly!**

$$\sum_R d^{abc} = \sum_R 2i \mathrm{tr}\{T^a [T^b T^c + T^c T^b]\}$$

- Determinant line bundle over $\mathfrak{U}$

$$\mathcal{O}^a \cap \mathcal{O}^b \qquad v^b_i(x) = \sum_l v^a_l(x) \tau(a \to b)_{li} \qquad g_{ab} \equiv \det \tau(a \to b)$$

$$\tilde{v}^a_i(x) = v^a_l(x) h^a_{lj} \qquad \tilde{g}_{ab} = \det h_a \, g_{ab} \det h_b^{-1}$$

$$\mathcal{O}^a \cap \mathcal{O}^b \cap \mathcal{O}^c \qquad g_{ac} = g_{ab} \, g_{bc}$$

Derive the Weyl fermion propergator

$$\langle \psi_L(x)\,\bar{\psi}_L(y)\rangle_F = \hat{P}_L D^{-1} P_R$$

By performing the grassman number integration over c_i, $\bar{c}_k$, one obtains

$$\langle \psi_L(x)\,\bar{\psi}_L(y)\rangle_F = \sum_{i,k} v_i(x)\left(M^{-1}\right)_{ik}\bar{v}_k(y), \qquad M_{ki} = (\bar{v}_k D v_i).$$

Noting

$$\hat{P}_L(x,y) = \sum_i v_i(x) v_i(y)^\dagger,\ \ P_R(x,y) = \sum_k \bar{v}_k(x)^\dagger \bar{v}_k(y),$$

one can show

$$\begin{aligned}&\sum_i (\bar{v}_k D v_i)(v_i^\dagger D^{-1} \bar{v}_l^\dagger)\\ &= \bar{v}_k D \hat{P}_L D^{-1} \bar{v}_l^\dagger = \bar{v}_k P_R D D^{-1} \bar{v}_l^\dagger = \delta_{kl},\end{aligned}$$

which implies

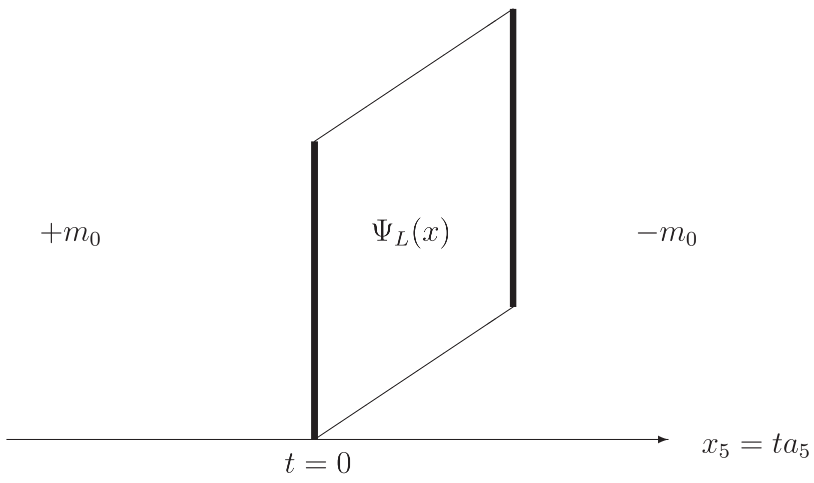
$$\left(M^{-1}\right)_{ik} = \sum_x v_i(x)^\dagger D^{-1} \bar{v}_k(x)^\dagger.$$

Then one obtains

$$
\begin{aligned}
\langle \psi_L(x)\, \bar{\psi}_L(y) \rangle_F &= \sum_{i,k} v_i(x) \left(M^{-1}\right)_{ik} \bar{v}_k(y) \\
&= \sum_{i,k} v_i(x)\, v_i(x)^\dagger D^{-1} \bar{v}_k(x)^\dagger\, \bar{v}_k(y) \\
&= \hat{P}_L D^{-1} P_R .
\end{aligned}
$$

Chiral fermion bound to domain-wall & Anomaly inflow

Callan-Harvey (1985)



5 dim. fermion coupled to Domain-wall

chiral mode bound to Domain-wall

$$\{i\gamma_\mu D_\mu + i\gamma_5 \partial_5 - m_0 \epsilon(x_5)\}\psi(x, x_5) = 0$$

$$V(x, x_5) = \gamma_5 m_0 \delta(x_5) \quad \Longrightarrow \quad \psi_0(x, x_5) \simeq \psi_-(x)\ \mathrm{e}^{-m_0|x_5|}$$

5 dim. Wilson fermion coupled to Domain-wall

Kaplan(1992)

$$S_{DW} = a^4 \sum_{x,t} \bar\psi(x,t) \left\{ a_5 D_\mathrm{w} - \frac{1-\gamma_5}{2}\partial_t - \frac{1+\gamma_5}{2}\partial_t{}^\dagger - m_0\, \epsilon(t+1/2) \right\} \psi(x,t)$$

5 dim. Wilson fermion -> Chern-Simons term
-> anomaly inflow

Kaplan-Jansen-Golterman (1993)

$$\pi\,\overline{\eta} \equiv \lim_{a_5\to 0}\lim_{T\to\infty} \mathrm{Im}\,\ln\,\det\left(D_{5\mathrm{w(T)}} - \frac{m_0}{a}\right)$$

cf. [*Aoyama-YK (1999)*]

$$\frac{d}{du}\,\overline{\eta}\,[U_\mu(x,t;u)] \;=\; \lim_{a_5\to 0}\lim_{T\to\infty}\frac{1}{\pi}\mathrm{ImTr}_{(T)}\,\frac{d}{du}D_{5\mathrm{w}(\infty)}\,\frac{1}{D_{5\mathrm{w}(\infty)}-\frac{m_0}{a}} + \frac{1}{\pi}\mathrm{Im}\,\mathrm{Tr}_x\,P_L\,\frac{d}{du}D\,\frac{1}{D}.$$

1st term in R.H.S. $\underset{a,a_5\to 0}{=} - \lim_{T'\to\infty}\int d^4x \int_{-T'}^{T'} dt\,\frac{1}{32\pi^2}\,\epsilon_{\mu MNKL}\mathrm{tr}\left\{\frac{d}{du}A_\mu\,F_{MN}F_{KL}\right\}(x,t;u)$

$$\begin{aligned} c \;&=\; \frac{1}{5!}\int_{-\pi}^{\pi}\frac{d^5k}{(2\pi)^5}\,\epsilon_{MNIJK}\mathrm{Tr}\left\{\left(\partial_M S^{-1}S\right)\left(\partial_N S^{-1}S\right)\times\right.\\ &\qquad\left.\left(\partial_I S^{-1}S\right)\left(\partial_J S^{-1}S\right)\left(\partial_K S^{-1}S\right)\right\}(k)\\ &=\; \frac{i}{8\pi^2}, \end{aligned}$$

$$T^{2n+1} \to S^{2n+1}$$

$$S^{-1}(k) = \sum_{M=1}^{5}\left(i\gamma_M \sin k_M + 2\sin^2\frac{k_M}{2}\right) - m_0 \quad (0 < m_0 < 2)$$

Chern-Simons term induced by Wilson-Dirac fermion in 2n+1 dim.

[*H. So (1984), Coste-Luscher (1989)*]

Lattice domain-wall fermion

[*Kaplan(1992)*] [*Shamir(1993)*]

$$S_{\rm DW} = \sum_{x,t} \bar{\psi}(x,t) X_{\rm w}^{(5)} \psi(x,t)$$

$$D_{\rm w} = -\gamma_\mu \frac{1}{2}(\nabla_\mu - \nabla_\mu^\dagger) + \frac{a}{2}\nabla_\mu \nabla_\mu^\dagger$$

$$\det X_{\mathrm{w}}^{(5)}|_{\mathrm{Dir.}} = \det D'_{\mathrm{ov}} \det X_{\mathrm{w}}^{(5)}|_{\mathrm{AP}} \quad (N \to \infty)$$

Master relations for d=2n+1, 2n+2 lattice domain wall fermions

$$\det X_{\rm w}^{(d)}\Big|_{\rm Dir}^{c_1} = \det\left(P_- + P_+ \prod_{t\in c_1} T_t\right)$$

$$\det X_{\rm w}^{(d)}\Big|_{\rm AP}^{c_1 c_2^{-1}} = \det\left(1 + \prod_{t\in c_1 c_2^{-1}} T_t\right)$$

$$T_t = \frac{1 - a_d H_t/2}{1 + a_d H_t/2}$$

$d = 2n+1$

$$H = \gamma_d X_{\rm w}^{(d-1)} \frac{1}{1 + a_d X_{\rm w}^{(d-1)}/2}$$

$d = 2n+2$

$$H = \tilde{\gamma}_d \tilde{X}_{\rm w}^{(d-1)} \frac{1}{1 + a_d \tilde{X}_{\rm w}^{(d-1)}/2} = \begin{pmatrix} 0 & X^{(d-1)} \\ X^{(d-1)\dagger} & 0 \end{pmatrix}$$

$$X^{(d-1)} = X_{\rm w}^{(d-1)} \frac{1}{1 + a_d X_{\rm w}^{(d-1)}/2}$$

$$\tilde{\gamma}_d = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = I \otimes \sigma_1, \quad \tilde{\gamma}_\mu = \gamma_\mu \otimes (-\sigma_3)$$

$$\det\left(D_{5\mathrm{w}} - m_0\right)_{[Dir]} = \det D_N \cdot \det\left(D_{5\mathrm{w}} - m_0\right)_{[AP]}$$

$$D_N = \frac{1}{2a}\left(1 + \gamma_5 \tanh\left(\frac{a_5 N H}{2}\right)\right)$$

$$H = -\frac{1}{a_5} \ln T$$

$$T = \begin{pmatrix} \frac{1}{B} & \frac{1}{B} a_5 C \\ -a_5 C^\dagger \frac{1}{B} & B + a_5^2 C^\dagger \frac{1}{B} C \end{pmatrix}$$

$$B = 1 + a_5(B_4 - m_0), \qquad D_\mathrm{w} = \begin{pmatrix} B_4 & -C \\ C^\dagger & -B_4 \end{pmatrix}$$

Evaluation of det $\left(D_{5\mathrm{w}} - \frac{m_0}{a}\right)$ $(a_5 = a = 1)$

$$
\begin{aligned}
& D_{5\mathrm{w}} - m_0 \\
= \; & (D_{\mathrm{w}} - m_0 + 1)\,\delta_{ts} - P_L \delta_{t+1,s} - P_R \delta_{t,s+1}
\end{aligned}
$$

$$
= \begin{pmatrix}
\begin{pmatrix} B & C \\ -C^\dagger & B \end{pmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} & \cdot & \cdot & \begin{bmatrix} +1 & 0 \\ 0 & 0 \end{bmatrix} \\
\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} B & C \\ -C^\dagger & B \end{pmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} & \cdot & \cdot \\
\cdot & \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} B & C \\ -C^\dagger & B \end{pmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} & \cdot \\
\cdot & \cdot & \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} B & C \\ -C^\dagger & B \end{pmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 \\ 0 & +1 \end{bmatrix} & \cdot & \cdot & \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} B & C \\ -C^\dagger & B \end{pmatrix}
\end{pmatrix}
$$

$$
\Rightarrow \begin{pmatrix}
\begin{pmatrix} C & B \\ B & -C^\dagger \end{pmatrix} & \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} & \cdot & \cdot & \begin{bmatrix} 0 & +1 \\ 0 & 0 \end{bmatrix} \\
\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} C & B \\ B & -C^\dagger \end{pmatrix} & \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} & \cdot & \cdot \\
\cdot & \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} C & B \\ B & -C^\dagger \end{pmatrix} & \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} & \cdot \\
\cdot & \cdot & \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} C & B \\ B & -C^\dagger \end{pmatrix} & \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 \\ +1 & 0 \end{bmatrix} & \cdot & \cdot & \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} & \begin{pmatrix} C & B \\ B & -C^\dagger \end{pmatrix}
\end{pmatrix}
$$

$$
\Rightarrow \begin{pmatrix}
\begin{pmatrix} B & 0 \\ -C^\dagger & -1 \end{pmatrix} & \cdot & \cdot & \cdot & \begin{bmatrix} +1 & C \\ 0 & B \end{bmatrix} \\
\begin{bmatrix} -1 & C \\ 0 & B \end{bmatrix} & \begin{pmatrix} B & 0 \\ -C^\dagger & -1 \end{pmatrix} & \cdot & \cdot & \cdot \\
\cdot & \begin{bmatrix} -1 & C \\ 0 & B \end{bmatrix} & \begin{pmatrix} B & 0 \\ -C^\dagger & -1 \end{pmatrix} & \cdot & \cdot \\
\cdot & \cdot & \begin{bmatrix} -1 & C \\ 0 & B \end{bmatrix} & \begin{pmatrix} B & 0 \\ -C^\dagger & -1 \end{pmatrix} & \cdot \\
\cdot & \cdot & \cdot & \begin{bmatrix} -1 & C \\ 0 & B \end{bmatrix} & \begin{pmatrix} B & 0 \\ -C^\dagger & +1 \end{pmatrix}
\end{pmatrix}
$$

Set

$$\alpha \equiv \begin{pmatrix} B & 0 \\ -C^\dagger & -1 \end{pmatrix} \quad \beta \equiv \begin{bmatrix} -1 & C \\ 0 & B \end{bmatrix}$$

$$\alpha_X \equiv \begin{pmatrix} B & 0 \\ -C^\dagger & X \end{pmatrix} \quad \beta_Y \equiv \begin{bmatrix} Y & C \\ 0 & B \end{bmatrix}$$

$X, Y = 0$ for Dirichlet B.C., $X, Y = +1$ for AP B.C.

Then

$$D_{5\mathrm{w}} - m_0 \;\Rightarrow\; \begin{pmatrix} \alpha & \cdot & \cdot & \cdot & \beta_Y \\ \beta & \alpha & \cdot & \cdot & \cdot \\ \cdot & \beta & \alpha & \cdot & \cdot \\ \cdot & \cdot & \beta & \alpha & \cdot \\ \cdot & \cdot & \cdot & \beta & \alpha_X \end{pmatrix} = \begin{pmatrix} \alpha & \cdot & \cdot & \cdot & \cdot \\ \beta & \alpha & \cdot & \cdot & \cdot \\ \cdot & \beta & \alpha & \cdot & \cdot \\ \cdot & \cdot & \beta & \alpha & \cdot \\ \cdot & \cdot & \cdot & \beta & \alpha_X \end{pmatrix} \times \begin{pmatrix} 1 & \cdot & \cdot & \cdot & -V_1 \\ \cdot & 1 & \cdot & \cdot & -V_2 \\ \cdot & \cdot & 1 & \cdot & -V_3 \\ \cdot & \cdot & \cdot & 1 & -V_4 \\ \cdot & \cdot & \cdot & \cdot & 1 - V_5 \end{pmatrix}$$

where

$$\begin{aligned} -\alpha V_1 &= \beta_Y \\ -\beta V_1 - \alpha V_2 &= 0 \\ -\beta V_2 - \alpha V_3 &= 0 \\ -\beta V_3 - \alpha V_4 &= 0 \\ -\beta V_4 + \alpha_X (1 - V_5) &= \alpha_X \end{aligned}$$

Now one can evaluate the determinant:

$$\det(D_{5\mathrm{w}} - m_0)_{X,Y} \;=\; \{\det\alpha\}^{N-1} \det\alpha_X \det(1 - V_N)$$

where

$$V_N = \alpha_X^{-1}\alpha \cdot \{-\alpha^{-1}\beta\}^N \cdot \beta^{-1}\beta_Y$$

$$\begin{aligned}
-\alpha^{-1}\beta &= \begin{pmatrix} \frac{1}{B} & -\frac{1}{B}C \\ -C^\dagger \frac{1}{B} & B + C^\dagger \frac{1}{B} C \end{pmatrix} = T = e^{-H} \\
\alpha^{-1}\alpha_Y &= \begin{pmatrix} 1 & 0 \\ 0 & -X \end{pmatrix} \\
\beta^{-1}\beta_Y &= \begin{pmatrix} -Y & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

For Dirichlet b.c. and Anti-Periodic b.c., respectively, one obtains

$$\det(D_{5\mathrm{w}} - m_0) \;=\; \{\det(P_L + P_R B)\}^N \cdot \det\gamma_5 \cdot \det(P_R + P_L T^N)$$

$$\det(D_{5\mathrm{w}} - m_0)_{AP} \;=\; \{\det(P_L + P_R B)\}^N \cdot \det\gamma_5 \cdot \det(1 + T^N)$$

$$\frac{P_R + P_L T^N}{1 + T^N} = \frac{1}{2}\left(1 + \gamma_5 \frac{1 - T^N}{1 + T^N}\right) = \frac{1}{2}\left(1 + \gamma_5 \tanh(NH/2)\right) = D_N$$

Block spin transformation & Extra dimensions/ d+1 Domain-wall fermion

Block spin trans.
$\equiv$Integration of high energy modes

Domain wall fermion
$\equiv$Integration of heavy modes

$\Downarrow$ $\Downarrow$

Local low energy effective action

$$S = \sum_x \bar{\psi}(x) D \psi(x)$$

$\Downarrow$

Ginsparg-Wilson rel.

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

References

[1] P. H. Ginsparg and K. G. Wilson, Phys. Rev. D **25**, 2649 (1982).

(I)

[2] H. Neuberger, Phys. Lett. B **417**, 141 (1998) [arXiv:hep-lat/9707022].

[3] H. Neuberger, Phys. Lett. B **427**, 353 (1998) [arXiv:hep-lat/9801031].

[4] M. Lüscher, Phys. Lett. B **428**, 342 (1998) [arXiv:hep-lat/9802011].

(II)

[5] P. Hernandez, K. Jansen and M. Lüscher, Nucl. Phys. B **552**, 363 (1999) [arXiv:hep-lat/9808010].

(III,IV)

[6] P. Hasenfratz, V. Laliena and F. Niedermayer, Phys. Lett. B **427**, 125 (1998) [arXiv:hep-lat/9801021].

(V)

[7] R. Narayanan and H. Neuberger, Nucl. Phys. B **412**, 574 (1994) [arXiv:hep-lat/9307006].

[8] R. Narayanan and H. Neuberger, Phys. Rev. Lett. **71**, 3251 (1993) [arXiv:hep-lat/9308011].

(VI)

[9] M. Lüscher, Nucl. Phys. B **549**, 295 (1999) [arXiv:hep-lat/9811032].

[10] M. Lüscher, Nucl. Phys. B **568**, 162 (2000) [arXiv:hep-lat/9904009].

(VII)

[11] S. Chandrasekharan, Phys. Rev. D **59**, 094502 (1999) [arXiv:hep-lat/9810007].

(VIII)

[12] D. B. Kaplan, Phys. Lett. B **288**, 342 (1992) [arXiv:hep-lat/9206013].

[13] Y. Shamir, Nucl. Phys. B **406**, 90 (1993) [arXiv:hep-lat/9303005].

[14] V. Furman and Y. Shamir, Nucl. Phys. B **439**, 54 (1995) [arXiv:hep-lat/9405004].

[15] H. Neuberger, Phys. Rev. D **57**, 5417 (1998) [arXiv:hep-lat/9710089].

[16] Y. Kikukawa and T. Noguchi, arXiv:hep-lat/9902022.

2. ハミルトニアン形式，Schwinger-Keldysh形式

Hamiltonian Approach to Lattice Gauge Theory *Kogut-Susskind (1975)*

$$H = H_G + H_F$$

$$H_G = a^3 \sum_{x\in\mathbb{L}^3} \left\{ \frac{g^2}{2} \sum_{k=1}^{3} \sum_{a} E_k^a(x) E_k^a(x) + \sum_{k,l=1}^{3} \frac{1}{2g^2} \mathrm{Tr}(1-U_{kl}(x))(1-U_{kl}(x))^\dagger \right\}$$

$$H_F = a^3 \sum_{x\in\mathbb{L}^3} \psi^\dagger(x)\gamma_0(D+m_0)\psi(x) \qquad D = D_{\mathrm{w}}^{(3)}, D_{\mathrm{ks}}^{(3)}, D_{\mathrm{ov}}^{(3)}$$

$$E_k^a(x) = \{T^a U_k(x)\}_{ij} \frac{\delta}{\delta\{U_k(x)\}_{ij}}$$

$$\left[E_k^a(x), \{U_{k'}(y)\}_{ij}\right] = +\{T^a U_k(x)\}_{ij}\, \delta_{k,k'}\, \delta_{x,y},$$

$$\left[E_k^a(x), \left\{U_{k'}^{-1}(y)\right\}_{ij}\right] = -\left\{U_k^{-1}(x) T^a\right\}_{ij}\, \delta_{k,k'}\, \delta_{x,y}.$$

$$[\psi(x), \psi^\dagger(x)] = \delta_{x,y}$$

$$G^a(x) = (\nabla_k{}^* E_k)^a(x) - \bar{\psi}(x) T^a \psi(x)$$

$$
\begin{aligned}
S_G &= \sum_{\tau,x} \sum_{\mu\nu} \frac{1}{2g^2} \mathrm{Tr}\{(1-U_{\mu\nu})(1-U_{\mu\nu})^\dagger\} \\
&= \sum_{\tau,x} \sum_{k} \frac{1}{g^2} \mathrm{Tr}\{2 - U_k(\tau,x)U_k^\dagger(\tau+1,x) - U_k^\dagger(\tau,x)U_k(\tau+1,x)\} \\
&\quad + \sum_{\tau,x} \sum_{kl} \frac{1}{2g^2} \mathrm{Tr}\{(1-U_{kl})(1-U_{kl})^\dagger\}(\tau,x)
\end{aligned}
$$

$$
\begin{aligned}
T_G(U,U') &= \mathrm{e}^{-\sum_x \sum_{kl} \frac{1}{4g^2} \mathrm{Tr}\left\{(1-U_{kl})(1-U_{kl})^\dagger\right\}} \times \\
&\qquad \mathrm{e}^{-\sum_x \sum_k \frac{1}{g^2} \mathrm{Tr}\left\{2-U_k U_k'^\dagger - U_k' U_k^\dagger\right\}} \times \\
&\qquad\qquad \mathrm{e}^{-\sum_x \sum_{kl} \frac{1}{4g^2} \mathrm{Tr}\left\{(1-U'_{kl})(1-U'_{kl})^\dagger\right\}} \\
&= \mathrm{e}^{-\sum_x \sum_{kl} \frac{1}{4g^2} \mathrm{Tr}\left\{(1-U_{kl})(1-U_{kl})^\dagger\right\}} \times \\
&\qquad \prod_x \prod_k \mathrm{e}^{-\frac{2N_c}{g^2}} \sum_r d_r \, L_r\left(2N_c/g^2\right) \mathrm{Tr}_r\{U_k U_k'^\dagger\} \times \\
&\qquad\qquad \mathrm{e}^{-\sum_x \sum_{kl} \frac{1}{4g^2} \mathrm{Tr}\left\{(1-U'_{kl})(1-U'_{kl})^\dagger\right\}} .
\end{aligned}
$$

$$
\mathrm{e}^{-\frac{1}{g^2} \mathrm{Tr}\left\{2-U-U^\dagger\right\}} = \mathrm{e}^{-\frac{2N_c}{g^2}} \sum_r d_r \, L_r\left(2N_c/g^2\right) \mathrm{Tr}_r\{U\} \qquad L_r(4/g^2) \to \mathrm{e}^{-*\frac{g^2}{2} C_2(r)} \quad (g \to 0)
$$

$$
\mathrm{e}^{-\frac{1}{g^2}\left\{1-\cos\theta\right\}} = \mathrm{e}^{-\frac{1}{g^2}} \sum_m I_m\left(1/g^2\right) \mathrm{e}^{im\theta} \qquad I_m(1/g^2) \to \mathrm{e}^{-\frac{g^2}{2} m^2} \quad (g \to 0)
$$

$$
\begin{aligned}
T_F &= B_t^{-1/2}\left[\beta_t(-\alpha_{t-1}^{-1})\right]B_{t-1}^{1/2}\\
&= B_t^{-1/2}[(2+a_0D_{3w})\gamma_0]_t(1-\mathcal{H}_t/2)\frac{1}{1+\mathcal{H}_{t-1}/2}\frac{1}{[(2+a_0D_{3w})\gamma_0]_{t-1}}B_{t-1}^{1/2}
\end{aligned}
$$

$$
\mathcal{H} = a_0\gamma_0(D_w^{(3)}+m_0)\frac{1}{1+a_0(D_w^{(3)}+m_0)/2}
$$

$$
T_F(\Delta) = B_t^{-1/2}[(2+a_0D_{3w})\gamma_0]_t(1-\Delta\mathcal{H}_t/2)\frac{1}{1+\Delta\mathcal{H}_{t-1}/2}\frac{1}{[(2+a_0D_{3w})\gamma_0]_{t-1}}B_{t-1}^{1/2}
$$

$$
T_F(\Delta)T_F(-\Delta) = 1 \qquad \Delta = \pm i
$$

Unitary Discrete Chiral Symmetry of Hamiltonian formalism in 1+1 dim.

— θ term & counter terms

- 1+1 dim. Schwinger model in Staggered fermion "representation"

$$H_{\mathrm{stg.}} = \sum_n \frac{g^2 a'}{2}\left\{\left(L_n + \frac{\theta}{2\pi}\right)^2 - \frac{1}{4}(-1)^n \bar{\chi}_n \chi_n\right\} - \sum_n \frac{i}{2a'}\left(\bar{\chi}_n U_n \chi_{n+1} - \bar{\chi}_{n+1} U_n^\dagger \chi_n\right)$$

$$\chi_n \to \chi'_n = U_n \chi_{n+1}$$

$$\bar{\chi}_n \to \bar{\chi}'_n = \bar{\chi}_{n+1} U_n^\dagger$$

[*Dempsey, Klebanov, Pufu, Zan 2023*)]

- 1+1 dim. Schwinger model in Wilson(Overlap)-Dirac fermion "representation"

$$H_0(g,\theta) = \sum_x \frac{g^2}{2}\left(E_1(x) + \frac{\theta}{2\pi}\right)^2 + \sum_x \psi^\dagger(x)\gamma_0 D_{\mathrm{w}}\psi(x),$$

$$H_{\mathrm{c}}(g,\theta) = \sum_x \frac{g^2}{2}\left\{\left(E_1(x) + \frac{\theta}{2\pi}\right)\left[\psi^\dagger(x+1)\left(\frac{1-\gamma_1}{2}\right)\psi(x+1) - 1\right] - \frac{1}{4}\left[\psi^\dagger(x+1)\left(\frac{1-\gamma_1}{2}\right)\psi(x+1) - 1\right]\right\},$$

[*Fujii. Fujikura, Okuda, Pedersen, Y.K.(2024)*]

$$\psi(x) = \begin{pmatrix} \chi_{2x} \\ \chi_{2x-1} \end{pmatrix}$$

$$\psi(x) \to \psi'(x) \equiv U^{-1}\psi(x)U = \gamma_5\left\{1 + \left(\frac{1-\gamma_1}{2}\right)D_1\right\}\psi(x)$$

$$\psi(x) \to \psi'(x) \equiv U^{-2}\psi(x)U^2 = U_1(x)\psi(x+\hat{1}).$$

$$D_{\mathrm{ov}}^{(1)} = D_{\mathrm{w}}/2$$

$$\Gamma = \gamma_5\left\{1 + \left(\frac{1-\gamma_1}{2}\right)\nabla_1\right\} = \gamma_5(1 - D_{\mathrm{ov}}^{(1)}) + i\gamma_0 D_{\mathrm{ov}}^{(1)}$$

Chiral Symmetry in the Wilson(Overlap)-Dirac fermion representation

$$Q_5 \equiv \sum_x \psi^\dagger(x)\gamma_5\left(1-\frac{D_{\rm w}}{2}\right)\psi(x).$$

$$[\psi(x), Q_5] = \gamma_5\left(1-\frac{D_{\rm w}}{2}\right)\psi(x),$$
$$[H_{\rm w}, Q_5] = 0,$$

Discrete Chiral Symmetry in the Wilson(Overlap)-Dirac fermion representation

$$\psi(x) \to \psi'(x) \equiv U^{-1}\psi(x)U = \gamma_5\left\{1+\left(\frac{1-\gamma_1}{2}\right)D_1\right\}\psi(x)$$

$$A = \gamma_5\left\{1+\left(\tfrac{1-\gamma_1}{2}\right)D_1\right\}$$

$$\begin{aligned}
A^\dagger A &= \left[\gamma_5\left\{1+\left(\frac{1-\gamma_1}{2}\right)D_1\right\}\right]^\dagger\gamma_5\left\{1+\left(\frac{1-\gamma_1}{2}\right)D_1\right\}\\
&= \left\{1+\left(\frac{1-\gamma_1}{2}\right)D_1^\dagger\right\}\left\{1+\left(\frac{1-\gamma_1}{2}\right)D_1\right\}\\
&= 1+\left(\frac{1-\gamma_1}{2}\right)D_1^\dagger+\left(\frac{1-\gamma_1}{2}\right)D_1+\left(\frac{1-\gamma_1}{2}\right)D_1^\dagger D_1\\
&= 1 \qquad (\because D_1^\dagger\times D_1 = -D_1-D_1^\dagger),
\end{aligned}$$

$$U = {\rm e}^{\sum_{x,y}\psi^\dagger(x)\{{\rm Ln}A\}_{xy}\psi(y)}, \qquad A_{xy} = \left\{\gamma_5\left[1+\left(\frac{1-\gamma_1}{2}\right)D_1\right]\right\}_{xy}$$

$$\psi(x) \to \psi'(x) \equiv U^{-2}\psi(x)U^2 = U_1(x)\psi(x+\hat{1}).$$

$$\psi(x) \to \psi'(x) \equiv U^{-1}\psi(x)U = \gamma_5 \left\{ 1 + \left(\frac{1-\gamma_1}{2} \right) D_1 \right\} \psi(x)$$

$$U = \mathrm{e}^{\sum_{x,y} \psi^\dagger(x)\{\mathrm{Ln}A\}_{xy}\psi(y)} \qquad A_{xy} = \gamma_5 \left\{ 1 + \left(\frac{1-\gamma_1}{2} \right) D_1 \right\}_{xy}$$

$$\psi(x) \to \psi'(x) \equiv U^{-2}\psi(x)U^2 = U_1(x)\psi(x+\hat{1}).$$

$$\begin{aligned} U^{-1}E_1(x)U &= E_1(x) + \psi^\dagger(x+1)\left(\frac{1-\gamma_1}{2}\right)\psi(x+1) && (1.43) \\ &= E_1(x+1) - \psi^\dagger(x+1)\left(\frac{1+\gamma_1}{2}\right)\psi(x+1) + 1 - G(x+1). \\ U^{-2}E_1(x)U^2 &= E_1(x) + \psi^\dagger(x+1)\psi(x+1) && (1.44) \\ &= E_1(x+1) + 1 - G(x+1). && (1.45) \\ U^{-1}\psi^\dagger(x)\left(\frac{1+\gamma_1}{2}\right)\psi(x)U &= \psi^\dagger(x+1)\left(\frac{1-\gamma_1}{2}\right)\psi(x+1), && (1.46) \\ U^{-1}\psi^\dagger(x)\left(\frac{1-\gamma_1}{2}\right)\psi(x)U &= \psi^\dagger(x)\left(\frac{1+\gamma_1}{2}\right)\psi(x), && (1.47) \\ U^{-1}G(x)U &= G(x). && (1.48) \end{aligned}$$

$$\begin{aligned}
E_1(x) - E_1(x-1) &= \psi^\dagger(x)\psi(x) - 1 + G(x) \\
L_{2x} - L_{2x-1} &= \psi^\dagger(x)\left(\frac{1+\gamma_1}{2}\right)\psi(x) + G_{2x} \\
L_{2x-1} - L_{2x-2} &= \psi^\dagger(x)\left(\frac{1-\gamma_1}{2}\right)\psi(x) - 1 + G_{2x-1} \\
\text{cf.}\quad L_n - L_{n-1} &= \chi_n^\dagger\chi_n - \frac{1-(-1)^n}{2} + G_n \qquad (n = 2x, 2x-1)
\end{aligned}$$

$$\begin{aligned}
E_1(x) &\equiv L_{2x} \\
U^{-1}E_1(x)U &= E_1(x) + \psi^\dagger(x+1)\left(\frac{1-\gamma_1}{2}\right)\psi(x+1) \\
&\equiv L_{2x+1} + 1 \\
U^{-2}E_1(x)U^2 &= E_1(x) + \psi^\dagger(x+1)\psi(x+1) \\
&= E_1(x+1) + 1 - G(x+1) \\
&\equiv L_{2x+2} + 1 - G(x+1)
\end{aligned}$$

$$\begin{aligned}
U^{-1}L_{2x}U &= L_{2x+1} + 1 \\
U^{-1}L_{2x+1}U &= L_{2x+2} - G(x+1) \\
\text{cf.}\quad U^{-1}L_nU &= L_{n+1} + \frac{1+(-1)^n}{2} - \frac{1-(-1)^n}{2}G(x) \qquad (n = 2x, 2x-1)
\end{aligned}$$

$$T(\theta) = \sum_x \frac{g^2}{2}\frac{1}{2}\left\{\left(L_{2x} + \frac{\theta}{2\pi}\right)^2 + \left(L_{2x-1} + \frac{\theta}{2\pi}\right)^2\right\}$$

$$\begin{aligned}
U^{-1}T(\theta)U &= \sum_x \frac{g^2}{2}\frac{1}{2}\left\{U^{-1}\left(L_{2x} + \frac{\theta}{2\pi}\right)^2 U + U^{-1}\left(L_{2x-1} + \frac{\theta}{2\pi}\right)^2 U\right\} \\
&= \sum_x \frac{g^2}{2}\frac{1}{2}\left\{\left(L_{2x+1} + \frac{\theta}{2\pi} + 1\right)^2 + \left(L_{2x} + \frac{\theta}{2\pi} - G(x)\right)^2\right\} \\
&= \sum_x \frac{g^2}{2}\frac{1}{2}\left\{\left(L_{2x+1} + \frac{\theta}{2\pi} + \frac{1}{2} + \frac{1}{2}\right)^2 + \left(L_{2x} + \frac{\theta}{2\pi} + \frac{1}{2} - \frac{1}{2}\right)^2\right\} \\
&= \sum_x \frac{g^2}{2}\frac{1}{2}\left\{\left(L_{2x+1} + \frac{\theta}{2\pi} + \frac{1}{2}\right)^2 + \left(L_{2x} + \frac{\theta}{2\pi} + \frac{1}{2}\right)^2\right. \\
&\qquad \left. + \left(L_{2x+1} + \frac{\theta}{2\pi} + \frac{1}{2}\right) - \left(L_{2x} + \frac{\theta}{2\pi} + \frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2\right\} \\
&= \sum_x \frac{g^2}{2}\frac{1}{2}\left\{\left(L_{2x+1} + \frac{\theta}{2\pi} + \frac{1}{2}\right)^2 + \left(L_{2x} + \frac{\theta}{2\pi} + \frac{1}{2}\right)^2\right. \\
&\qquad \left. + \left(\psi^\dagger(x+1)\left(\frac{1-\gamma_1}{2}\right)\psi(x+1) - 1\right) + \frac{1}{2}\right\} \\
&= \sum_x \frac{g^2}{2}\frac{1}{2}\left\{\left(L_{2x+1} + \frac{\theta}{2\pi} + \frac{1}{2}\right)^2 + \left(L_{2x} + \frac{\theta}{2\pi} + \frac{1}{2}\right)^2\right. \\
&\qquad \left. + \frac{1}{2}\left(\psi^\dagger(x)\left(\frac{1-\gamma_1}{2}\right)\psi(x) - \psi^\dagger(x)\left(\frac{1+\gamma_1}{2}\right)\psi(x)\right) - \frac{1}{2}G(x)\right\} \\
&= T(\theta+\pi) + X + \sum_x \frac{g^2}{2}\left(E_1(x) + \frac{\theta}{2\pi}\right)G(x) + \sum_x \frac{g^2}{4}G(x)
\end{aligned}$$

$$
\begin{aligned}
X &= \sum_x \frac{g^2}{4} \left\{ \left(\psi^\dagger(x) \left(\frac{1-\gamma_1}{2} \right) \psi(x) - 1 \right) + \frac{1}{2} \right\} \\
&= \sum_x \frac{g^2}{8} \left(\psi^\dagger(x) \left(\frac{1-\gamma_1}{2} \right) \psi(x) - \psi^\dagger(x) \left(\frac{1+\gamma_1}{2} \right) \psi(x) \right) - \sum_x \frac{g^2}{8} G(x)
\end{aligned}
$$

$$
U^{-1} X U = -X - \sum_x \frac{g^2}{4} G(x)
$$

$$
\begin{aligned}
\Delta T &\equiv \frac{1}{2} X \\
U^{-1} \Delta T U &= -\Delta T - \sum_x \frac{g^2}{8} G(x) \\
&= \Delta T - X - \sum_x \frac{g^2}{8} G(x)
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad & U^{-1} [T(\theta) + \Delta T] U \\
&= [T(\theta + \pi) + \Delta T] - \sum_x \frac{g^2}{8} G(x) + \sum_x \frac{g^2}{2} \left(E_1(x) + \frac{\theta}{2\pi} \right) G(x) + \sum_x \frac{g^2}{4} G(x)
\end{aligned}
$$

$$H_{\mathrm{stg.}} = \quad H_{\mathrm{w}} + \sum_{x} \frac{g^2(2a')}{2} \left\{ \frac{1}{2} - \frac{1}{4} G_{2x} \right\}$$

$$2a' = a, \qquad \chi_{2x} = b(x), \quad \chi_{2x-1} = d^{\dagger}(x) \qquad (\gamma_0 = \sigma_2 \to -\sigma_2)$$

- 1+1 dim. Schwinger model in 2+1 DWF fermion "representation" ???

$$H_F = \sum_x \sum_{t=0}^{N_t} \bar{\psi}(x,t)\gamma_0(D_{\rm w}^{(2)} - m_0)\psi(x,t) \qquad \left(\frac{1+\gamma_3}{2}\right)\psi(x,N_t+1)=0 \qquad \left(\frac{1-\gamma_3}{2}\right)\psi(x,-1)=0$$

Relation to Wilson(Overlap)-Dirac fermion "representation"

$$\frac{\mathrm{Det}(D_{\rm w}^{(2)} - m_0)|_{\rm Dir.}}{\mathrm{Det}(D_{\rm w}^{(2)} - m_0)|_{\rm AP.}} = \mathrm{Det} D'_{\rm ov} \simeq \mathrm{Det} D_{\rm w}$$

Relation to Stacy (Tangent) fermion

$$\begin{aligned} D_c^{-1} &= D_{\rm ov}^{-1} - 1 = 2D_{\rm w}^{-1} - 1 \\ &= -i\gamma_1 \frac{\sin p}{1-\cos p} = -i\gamma_1 \frac{\sin(p/2)}{\cos(p/2)} \end{aligned}$$

$$D_c = i\gamma_1 \tan(p/2)$$

[*Haegeman et. al (2024)*]

$$i\partial_t N|\Psi\rangle = \tilde{H}|\Psi\rangle \qquad \tilde{H} = \gamma_0(i\gamma_1 \sin p)$$

$$\tilde{N} = \frac{1-\cos p}{2} \qquad \{\chi(p), \chi^\dagger(p)\} = \frac{1-\cos p}{2}$$

Unitary Discrete Chiral Symmetry of Hamiltonian formalism in 3+1 dim.

$$H = a^3 \sum_{x \in \mathbb{L}^3} \frac{g^2}{2} \sum_{k=1}^{3} E_k^a(x) E_k^a(x) + a^3 \sum_{x \in \mathbb{L}^3} \bar{\psi}(x) \gamma_0 D \psi(x)$$

$$\mathcal{H} \equiv \gamma_0 D,$$

[*Creutz, Horvath, Neuberger (2001)*]

$$D \equiv D_{\mathrm{ov}}^{(3)} = \frac{1}{2}\left(1 + X_{\mathrm{w}}^{(3)} \frac{1}{\sqrt{X_{\mathrm{w}}^{(3)\dagger} X_{\mathrm{w}}^{(3)}}}\right)$$

[*Neuberger, Y.K. (1998)*]

$$D^\dagger = \gamma_5 D \gamma_5 = \gamma_0 D \gamma_0$$

$$D + D^\dagger = 2 D^\dagger D = 2 D D^\dagger$$

$$\mathcal{Q}_5 = \gamma_5 (1 - D), \qquad \left[\mathcal{Q}_5, \mathcal{H}\right] = 0, \qquad \mathcal{Q}_5^2 + \mathcal{H}^2 = 1$$

$$\Gamma_5 = \gamma_5 (1 - D) + i \gamma_0 D \qquad \Gamma_5^\dagger \Gamma_5 = 1$$

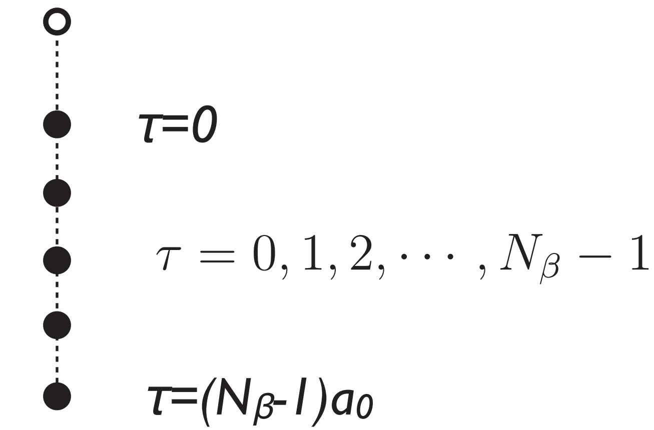
θ term & counter terms ?

Schwinger-Keldysh formalism for LFT

$$\langle \hat{O}(\tau)\rangle_{\beta,\mu} \equiv \mathrm{Tr}\left[\{\hat{T}_{+1}\}^{\beta-\tau}\,\hat{O}\,\{\hat{T}_{+1}\}^{\tau}\right]/Z(\beta,\mu)$$

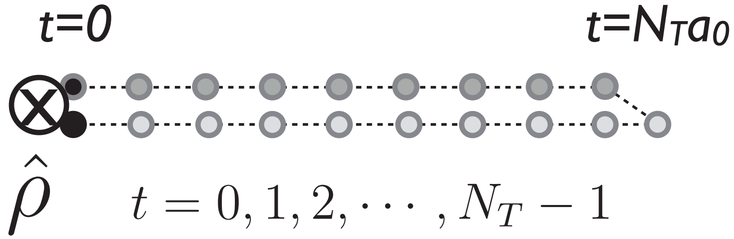
$$\hat{T}_{+1} = \mathrm{e}^{-a_0\hat{H}} \quad \left(= \mathrm{e}^{-a_0\hat{V}/2}\mathrm{e}^{-a_0\hat{\Pi}^2/2}\mathrm{e}^{-a_0\hat{V}/2}\right)$$

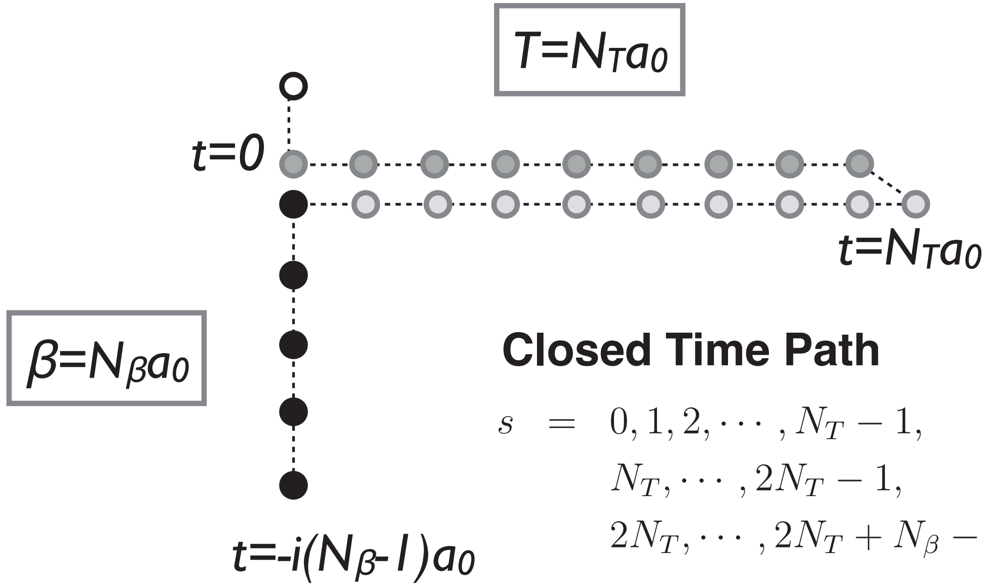
$$\Longrightarrow \quad \hat{T}_{\pm i} = \mathrm{e}^{\mp i a_0\hat{H}}$$



$$\langle \hat{O}(t)\rangle \equiv \mathrm{Tr}\left[\hat{\rho}\,\{\hat{T}_{-i}\}^{T}\,\{\hat{T}_{+i}\}^{T-t}\,\hat{O}\,\{\hat{T}_{+i}\}^{t}\right]/Z$$

$$\langle \hat{O}(t)\rangle \equiv \mathrm{Tr}\left[\{\hat{T}_{+1}\}^{\beta}\,\{\hat{T}_{-i}\}^{T}\,\{\hat{T}_{+i}\}^{T-t}\,\hat{O}\,\{\hat{T}_{+i}\}^{t}\right]/Z$$





Closed Time Path

$$s = 0, 1, 2, \cdots, N_T - 1,\quad N_T, \cdots, 2N_T - 1,\quad 2N_T, \cdots, 2N_T + N_\beta - 1$$

$$\Delta_s = \begin{cases} +i + \epsilon \\ -i + \epsilon \\ +1 \end{cases}$$

Construct Transfer matrixes for Scalar, Link and Wilson fermion fields so that

$$\hat{T}_{-i}\hat{T}_{+i} = \boldsymbol{I}, \quad \hat{T}_{\pm i} = \hat{A}\,\hat{U}_{\pm i}\,\hat{A}^{-1}$$
$$\hat{T}_{+1}\hat{T}_{\pm i} - \hat{T}_{\pm i}\hat{T}_{+1} \neq 0$$

* to recover in the continuum limit

[*Alexandru et al.(2016)*] [*Fujii, Hoshina & YK(2019)*]

Schwinger-Keldysh formalism for Lattice Gauge Theory(QCD)

[Fujii, Hoshina & YK]

$$U_0(x,s)=1\,(s=0,\cdots,2N_T+N_\beta-2),\;\; U_0(x,2N_T+N_\beta-1)=P(x)$$

$$S_G^{[\Delta]}=\sum_{s,x}\left\{\sum_k K_{\Delta_s}\left[U_k(s)U_k{}^\dagger(s-1)\right]+\sum_{kl}\frac{\Delta_s+\Delta_{s-1}}{2g^2}\mathrm{ReTr}(1-U_{kl})(s)\right\}$$

$$e^{-K_\Delta\left[UU'^\dagger\right]}=\begin{cases} e^{-\frac{1}{g^2}\mathrm{Tr}\left\{2-UU'^\dagger-U'U^\dagger\right\}} & (\Delta=+1)\\ e^{-\frac{2N_c}{g^2\Delta}}\;\sum_r d_r\left\{L_r\left(2N_c/g^2\right)\right\}^\Delta \mathrm{Tr}_r\{UU'^\dagger\} & (\Delta=\pm i)\end{cases}$$

$$\Delta=\pm i+\epsilon$$

$$S_W^{[\Delta]}=\sum_{s,s',x}\bar\psi(s,x)\left\{-\left(\frac{1-\gamma_0}{2}\right)\delta_{s+1,s'}\,e^{-\mu}-\left(\frac{1+\gamma_0}{2}\right)\delta_{s,s'+1}\,e^{+\mu}+\delta_{ss'}\right.$$
$$\left.+a_0\left[\gamma_k\frac{1}{2}\left(\nabla_k-\nabla_k^\dagger\right)+\frac{1}{2}\nabla_k\nabla_k^\dagger+m\right]V_{s,s'}\right\}\psi(s',x)$$

[Kanwar, Wagman]
cf. [Matsumoto]

$$V_{s,s'}=\frac{\Delta_s-1}{2}\left(\frac{1-\gamma_0}{2}\right)\delta_{s,s'-1}+\frac{\Delta_{s-1}-1}{2}\left(\frac{1+\gamma_0}{2}\right)\delta_{s,s'+1}$$
$$+\left[\frac{\Delta_s+1}{2}\left(\frac{1+\gamma_0}{2}\right)+\frac{\Delta_{s+1}-1}{2}\left(\frac{1-\gamma_0}{2}\right)\right]\delta_{s,s'}$$

Closed Time Path

$$s=0,1,2,\cdots,N_T-1,\; N_T,\cdots,2N_T-1,\; 2N_T,\cdots,2N_T+N_\beta-1$$

$$\Delta_s=\begin{cases}+i+\epsilon\\ -i+\epsilon\\ +1\end{cases}$$

$$\hat{T}_\Delta = \hat{T}_{G\Delta} \otimes \hat{T}_{F\Delta}$$

$$\hat{T}_{G\Delta} = \prod_x \left\{ \mathrm{e}^{-\sum_{kl} \frac{\Delta_s}{2g^2} \mathrm{ReTr}(1-U_{kl})(s)} \mathrm{e}^{-\sum_k K_{\Delta_s}\left[U(s)U(s-1)^{-1}\right]} \mathrm{e}^{-\sum_{kl} \frac{\Delta_{s-1}}{2g^2} \mathrm{ReTr}(1-U_{kl})(s-1)} \right\}$$

$$\mathrm{e}^{-K_\Delta\left[UU'^\dagger\right]} = \begin{cases} \mathrm{e}^{-\frac{1}{g^2}\mathrm{Tr}\left\{2-UU'^\dagger-U'U^\dagger\right\}} & (\Delta = +1) \\ \mathrm{e}^{-\frac{2N_c}{g^2\Delta}} \sum_r d_r \left\{L_r\left(2N_c/g^2\right)\right\}^\Delta \mathrm{Tr}_r\{UU'^\dagger\} & (\Delta = \pm i) \end{cases}$$

$\Delta = \pm i + \epsilon$

$$\hat{T}_{F\Delta} = A_s \left(1 - \Delta\mathcal{H}_s/2\right) \frac{1}{1 + \Delta\mathcal{H}_{s-1}/2} A_{s-1}^{-1}$$

$$\mathcal{H}_s/2 = \gamma_0 a_0 D_{3w} \frac{1}{(2 + a_0 D_{3w})_s} \qquad A_s = B_s^{-1/2}[(2 + a_0 D_{3w})\gamma_0]_s$$

$$B_s = 1 + a_0[\frac{1}{2}\nabla_k \nabla_k^\dagger + m]_s$$

$$\begin{aligned} Z_{\mathrm{latQCD}}[\beta, \mu; T] &= \mathrm{Tr}\left[\left\{\hat{T}_{+1}\right\}^\beta \left\{\hat{T}_{-i}\right\}^T \left\{\hat{T}_{-i}\right\}^T\right] \\ &= \int \mathcal{D}[U]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\, \mathrm{e}^{-S_G^{[\Delta]}[U]-S_W^{[\Delta]}[\psi,\bar{\psi},\mu]} \prod_{s=1}^{2N_T} \{\det \alpha_{s\Delta_s}\}^{-1} \end{aligned}$$

$$\alpha_{s\Delta} = \frac{1}{2}[\Delta\, a_0 D_{3w} + (2 + a_0 D_{3w})\gamma_0]_s$$

- Spectral function $p_0 \in \left[-\frac{\pi}{a_0}, +\frac{\pi}{a_0}\right]$ $T \to \infty$

$$\rho(p,\beta)_T \equiv \sum_{t=-T/2}^{T/2} \sum_{x} \mathrm{e}^{ip_0 t - i\boldsymbol{p}\boldsymbol{x}} \left\langle \left[\phi(t+T/2,\boldsymbol{x}), \phi(T/2,\boldsymbol{0})\right] \right\rangle_\beta$$

- Responce to time-dependent external source

ex. EM field (U(1) link field) $V_\mu(t,\boldsymbol{x}) = \mathrm{e}^{ie_0 A_\mu(t,\boldsymbol{x})} \Longleftrightarrow J_\mu(t,\boldsymbol{x})$

conserved U(1) current of the lattice fermions

$$\langle \bar{J}_k(t,\boldsymbol{x})\rangle_\beta \,[V_0]$$

$$\langle \bar{J}_k(t,\boldsymbol{x})\rangle_\beta \,[V_0] \simeq \sum_{t'\boldsymbol{x}'} G_R(t,\boldsymbol{x};t',\boldsymbol{x}') A_0(t',\boldsymbol{x}')$$

$$G_R(x,x') = i\theta(t-t') \left\langle \left[\bar{J}_k(t,\boldsymbol{x}), \bar{J}_0(t',\boldsymbol{x}')\right] \right\rangle_\beta$$

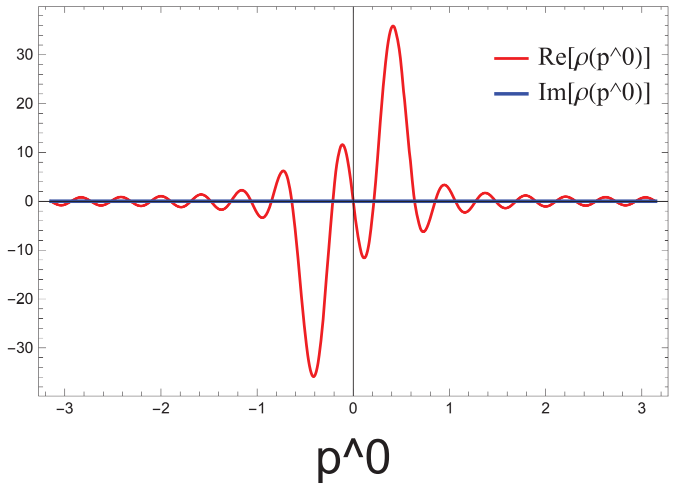
- Conductivity & Kubo's responce function

$$\sigma = \frac{1}{d} \sum_{t=t_0'}^{T} \sum_{s'=t_0'}^{-N_\beta - t_0'} \sum_{\boldsymbol{x}} \left\langle \left[\bar{J}_k(t,\boldsymbol{x}), \bar{J}_k(s',\boldsymbol{x}')\right] \right\rangle_\beta$$

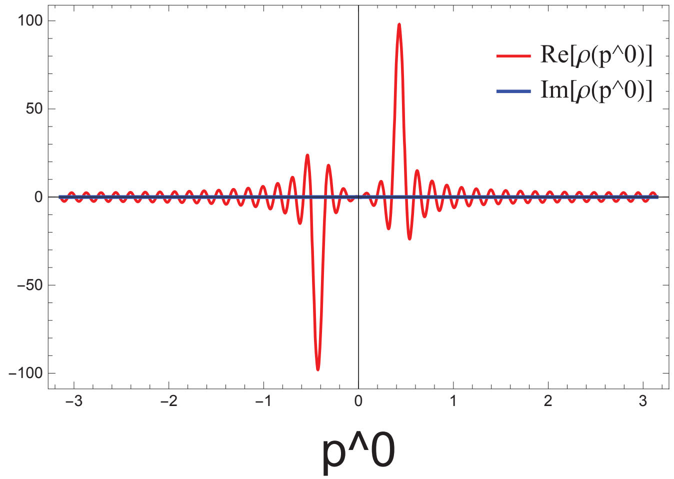
$$G^K_{kl}(x,x') = \frac{1}{N_\beta} \sum_{s'=t'}^{-N_\beta - t'} \sum_{\boldsymbol{x}} \left\langle \left[\bar{J}_k(t,\boldsymbol{x}), \bar{J}_l(s',\boldsymbol{x}')\right] \right\rangle_\beta$$

[Fujii, Hoshina & YK (2019)]

- 実際に有限格子サイズ (N_T を有限) でプロット
- N_T を大きくすると、よりデルタ関数的になる



(a) $N_T = 30$



(b) $N_T = 80$

$$a_0 = 1, N_\beta = 8, \ \hat{m}_0 = 0.3, \ \hat{p}_1 = 0.3$$

$$\rho(p; N_T, N_\beta, a_0) = \frac{1}{2\sin E} \sum_{n'_t=-N_T/2}^{N_T/2} \left(\cos(p^0 - E)n'_t - \cos(p^0 + E)n'_t\right)$$

3. 格子フェルミオンの 't HooftアノマリーとアノマリーInflow

Domain-wall fermion and 4D TI/TSC & SPT phase

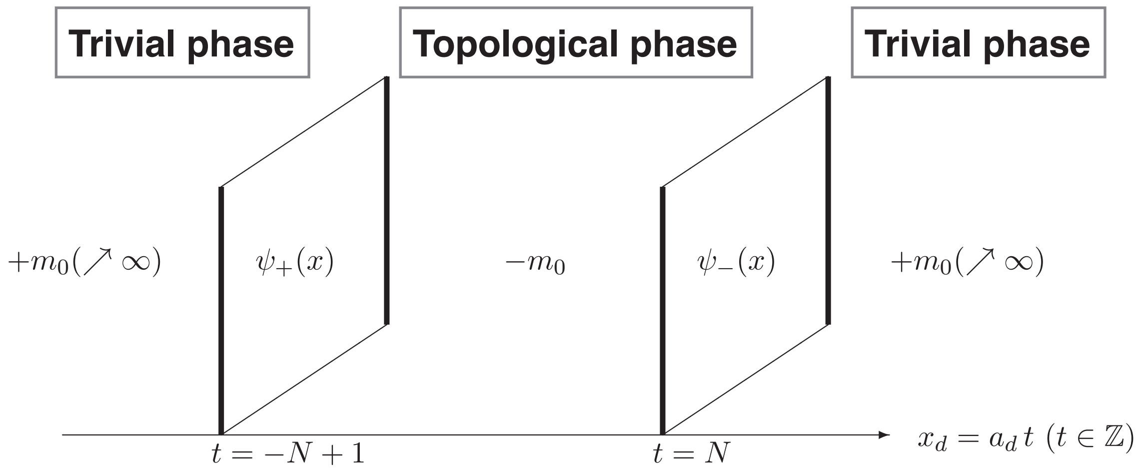
- Hamiltonian formalism of 4+1 dim. lattice DWF

= Free Fermion 4D TI/TSC (with Time-reversal symmetry, cf. 2+1dim. IQHE)

[*Creutz, Horvath(1994)*] [*Qi, Hughes, Zhang (2008)*]
[*Wen(2013), You-BenTov-Xu(2014),You-Xu (2015),Wang-Wen (2018)*]

$$\hat{H}_{4\mathrm{DTI}} = \sum_{i=1}^{\nu} \sum_{p} \hat{a}_i(p)^\dagger \Big\{ \sum_{k=1}^{4} \alpha_k \sin(p_k) + \beta \Big(\big[\sum_{k=1}^{4} \cos(p_k) - 4 \big] + m \Big) \Big\} \hat{a}_i(p)$$

$$\hat{H}^{(\mathrm{bd})}_{3\mathrm{D}} = \sum_{i=1}^{\nu} \int d^3x \, \hat{\psi}_i(x)^\dagger \Big\{ \sum_{l=1}^{3} (-i)\sigma_l \partial_l \Big\} \hat{\psi}_i(x)$$



- Edge chiral modes are described by low energy effective 3D Hamiltonian
- 3D Edge chiral modes described directly on the lattice by
  low energy effective 3+1dim. lattice model of Overlap Weyl fermions

Relation to 4D TI/TSC and SPT phase

- classification of 4D TI/TSC, SPT phase

free fermion case (K theory):

- TI (with U(1)) AII type classified by v in $\mathbb{Z}$
- TSC (without U(1), Majorana mass) DIII type with only a trivial vacuum

interacting case (cobordism):

- v=16, **16** of SO(10), ~~Majorana mass~~, U(1) broken by interaction

$\Omega^{\text{spin}}_5(B\text{Spin}(10)) = 0$ [*Garcia-Etxebarria & Montero (2018), Wang-Wen-Witten(2018)*]

(for 16 of SO(9), one can breaks U(1) by Majorana mass —> "DIII classified by 0"

if U(1) breaks to $\mathbb{Z}_4$, spin $\mathbb{Z}_4$ str. —> "DIII classified by $\mathbb{Z}_{16}$")

- "AII" is trivialized by a certain SO(10)-invariant and U(1)-breaking interaction, and the boundary edge modes may be gapped completely without symmetry breaking
- No obstruction for SO(9) => SO(10) symmetry restoration: $\Pi_d(S^9) = 0$ (d =0, …,9)

$$\lambda \left[\psi^T(x) i\gamma_5 c_d C\Gamma^a \psi(x)\right]^2 \Leftrightarrow y \left[\psi^T(x) i\gamma_5 c_d C\Gamma^a \psi(x)\right] E^a(x) \quad E^a(x)E^a(x) = 1$$

[*Wen(2013)*]

[*Morimoto, Furusaki, Mudry (2015)*]

[*Eichten-Preskill(1986)*]

free fermion case (K theory):

Cartan	d												
	0	1	2	3	4	5	6	7	8	9	10	11	...
Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	...
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	...
Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	...
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	...
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	...
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	...
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	...
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	...
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	...

Cartan label	T	C	S	Hamiltonian	G/H (ferm. NLσM)
A (unitary)	0	0	0	$\mathrm{U}(N)$	$\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$
AI (orthogonal)	+1	0	0	$\mathrm{U}(N)/\mathrm{O}(N)$	$\mathrm{Sp}(2n)/\mathrm{Sp}(n)\times\mathrm{Sp}(n)$
AII (symplectic)	−1	0	0	$\mathrm{U}(2N)/\mathrm{Sp}(2N)$	$\mathrm{O}(2n)/\mathrm{O}(n)\times\mathrm{O}(n)$
AIII (ch. unit.)	0	0	1	$\mathrm{U}(N+M)/\mathrm{U}(N)\times\mathrm{U}(M)$	$\mathrm{U}(n)$
BDI (ch. orth.)	+1	+1	1	$\mathrm{O}(N+M)/\mathrm{O}(N)\times\mathrm{O}(M)$	$\mathrm{U}(2n)/\mathrm{Sp}(2n)$
CII (ch. sympl.)	−1	−1	1	$\mathrm{Sp}(N+M)/\mathrm{Sp}(N)\times\mathrm{Sp}(M)$	$\mathrm{U}(2n)/\mathrm{O}(2n)$
D (BdG)	0	+1	0	$\mathrm{SO}(2N)$	$\mathrm{O}(2n)/\mathrm{U}(n)$
C (BdG)	0	−1	0	$\mathrm{Sp}(2N)$	$\mathrm{Sp}(2n)/\mathrm{U}(n)$
DIII (BdG)	−1	+1	1	$\mathrm{SO}(2N)/\mathrm{U}(N)$	$\mathrm{O}(2n)$
CI (BdG)	+1	−1	1	$\mathrm{Sp}(2N)/\mathrm{U}(N)$	$\mathrm{Sp}(2n)$

$$\mathcal{T}: \quad U_T^\dagger\, \mathcal{H}^*\, U_T = +\mathcal{H}$$

$$\mathcal{C}: \quad U_C^\dagger\, \mathcal{H}^*\, U_C = -\mathcal{H}$$

$$\mathcal{H}^* = \mathcal{H}^T$$

$$\mathcal{S} = \mathcal{T}\cdot\mathcal{C}$$

interacting case (cobordism):

Cartan	d = 0	1	2	3	4	5	6	7	8	9	10	11	...
Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}\times\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	...
AIII	0	$\mathbb{Z}_4$	0	$\mathbb{Z}_8\times\mathbb{Z}_2$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	...
Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	...
BDI	$\mathbb{Z}_2$	$\mathbb{Z}_8$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	...
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	...
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	...
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	...
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	...
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	...

SPT相

可逆な場の理論(invertible field theory)　ヒルベルト空間が1次元であるような場の理論

d次元の(reflection positiveで)可逆な場の理論の連続変形の同値類.

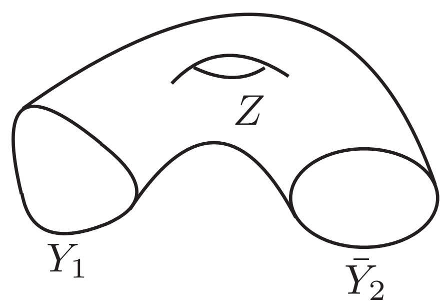
<−>　$d+1$次のボルディズム群のアンダーソン双対の元.

$$\Omega_d^S(BG)_{\mathrm{tor}} \times \Omega_{d+1}^S(BG)_{\mathrm{free}}$$

ポントリャーギン双対

$$\mathrm{Hom}\left(\Omega_d^S(BG), U(1)\right)$$

ボルディズム(bordism)群



$Y_1 \sqcup \bar{Y}_2$

Y_1 and Y_2 are bordant

$Y_1 \sim Y_2$

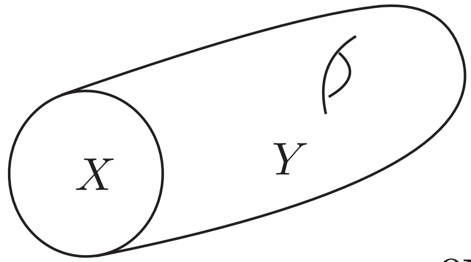
an abelian group

$[Y_1] + [Y_2] = [Y_1 \sqcup Y_2]$

"Dai-Freed anomalies" [*Garcia-Etxebarria & Montero (2018), Wang-Wen-Witten(2018)*]

$$Z[A] = \int [D\psi] e^{-S_{\text{fermion}}(\psi, A)}$$

$$Z[A] = \det(i\not{D})$$



$$\exp\left(2\pi i \eta_Y\right)$$

Atiyah-Patodi-Singer (APS) η-invariant

$$\eta_Y = \frac{1}{2}\left(\sum_{\lambda \neq 0} \text{sign}(\lambda) + \dim\ker(i\mathcal{D}_Y)\right)_{\text{reg.}}$$

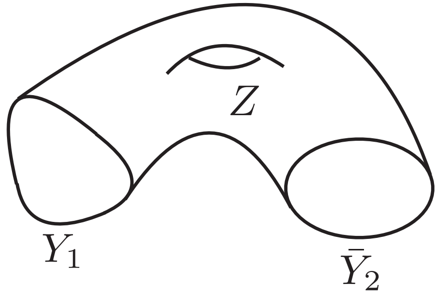
$$i\not{D}_Y = i\gamma^\tau \left(\frac{\partial}{\partial\tau} + i\not{D}'_X\right)$$

$$i\not{D}'_X = \begin{pmatrix} 0 & i\not{D}_X \\ -i\not{D}^\dagger_X & 0 \end{pmatrix} \qquad \gamma^\tau = \text{diag}\left(\mathbf{I}_d, -\mathbf{I}_d\right)$$

$\exp\left(2\pi i \eta_Y\right)$

bordism invariant

a group homomorphism from $\Omega^{\mathrm{Spin}}_{d+1}(BG)$ to $U(1)$



$Y_1 \sqcup \bar{Y}_2$

Y_1 and Y_2 are bordant

$Y_1 \sim Y_2$

an abelian group

$[Y_1] + [Y_2] = [Y_1 \sqcup Y_2]$

$$\exp(2\pi i\, \eta_{\overline{Y}}) = \exp(-2\pi i\, \eta_Y)$$

$$\exp(2\pi i\, \eta_{Y_1 \sqcup \overline{Y_2}}) = \frac{\exp(2\pi i\, \eta_{Y_1})}{\exp(2\pi i\, \eta_{Y_2})}$$

$$\frac{\exp(2\pi i\, \eta_{Y_1})}{\exp(2\pi i\, \eta_{Y_2})} = \exp(2\pi i\, \eta_{Y_1 \sqcup \overline{Y_2}}) = \exp(2\pi i \int_Z I_{d+2}) = 1$$

$$\mathrm{Ind}(\not{D}_Z) = \eta_Y + \int_Z \hat{A}(R)\, \mathrm{ch}(F)$$

APS index theorem

$$\exp(2\pi i\, \eta_Y) = \exp\left(2\pi i \int_Z \hat{A}(R)\, \mathrm{ch}(F)\right) = \exp\left(2\pi i \int_Z I_{d+2}\right)$$

“Dai-Freed anomalies”

- Compute $\Omega^{\mathrm{Spin}}_{d+1}(BG)$. If it vanishes, there can be no Dai-Freed anomaly.
- If $\Omega^{\mathrm{Spin}}_{d+1}(BG) \neq 0$, compute $\exp(2\pi i\,\eta)\colon \Omega^{\mathrm{Spin}}_{d+1}(BG) \to U(1)$, typically by explicit computation on convenient generators of the bordism group.

$$\Omega^{\mathrm{Spin}}_5(BSpin(n)) = 0 \qquad \text{for } n \geq 8$$

$$\Omega^{\mathrm{Spin}}_5(BSU(n)) = 0 \qquad \text{for } \quad n > 2$$

$$\Omega^{\mathrm{Pin}^+}_4 \approx \Omega^{\mathrm{Spin}^{\mathbb{Z}_4}}_5$$

$\mathrm{Spin}^{\mathbb{Z}_4}$ structure

[*Tachikawa & Yonekura (2018)*]

$(\mathrm{Spin} \times \mathbb{Z}_4)/\mathbb{Z}_2$ $\quad (-1)^F \quad \mathbb{Z}_2$ subgroup of $\mathbb{Z}_4$ are identified

$\Omega_4^{\mathrm{Pin}^+} \approx \Omega_5^{\mathrm{Spin}^{\mathbb{Z}_4}}$ $\quad \mathrm{Spin}^{\mathbb{Z}_4}$ bundles which contain domain walls on which 3d Pin^+ fermions localize

For each 4d Weyl fermion with charge 1 modulo 4,

one 3d Pin^+ Majorana fermion **"DIII classified by $\mathbb{Z}_{16}$"**

the $\mathbb{Z}_4$ group $\quad$ center of $Spin(10)$

$X \equiv -2Y + 5(B-L)$

$X = 4k+1$

SM field	**SU(3)**	**SU(2)**	**Y**	**B – L**	**X**
l_L^c	**1**	**2**	-3	3	21
q_L^c	$\bar{\mathbf{3}}$	**2**	1	-1	-7
l_R	**1**	**1**	6	-3	-27
u_R	**3**	**1**	-4	1	13
d_R	**3**	**1**	2	1	1
ν_R	**1**	**1**	0	-3	-15
H	**1**	**2**	3	0	-6

Dai-Freed theorem on the lattice

[Aoyama-YK (1999)] [YK (2002)]
[Pedersen-YK (2019)]

$$\lim_{N\to\infty} \frac{\det X_{\rm w}^{(5)}|^{c}_{\rm Dir.}}{\sqrt{\det X_{\rm w}^{(5)}|^{c\cdot c^{-1}}_{\rm AP}}}$$

boundary contribution (overlap Weyl fermions)

bulk contribution ($\mathrm{DW}|_{\mathrm{Dir}}$)

- same basis-vector dependence

=> same U(1) bundle over $\mathfrak{U} = \Big\{ \{U(x,\mu)\} \Big| \parallel 1 - U_{\mu\nu}(x) \parallel < \epsilon \ {}^\forall(x,\mu,\nu) \Big\}$

cf. [Dai-Freed (1994)]

[Yonekura (2016)]

- connection of the U(1) bundle

$$\nabla_\eta \, \mathrm{e}^{-i2\pi\hat{\eta}_{\rm DF}(\mathbb{Y}|_{\rm Dir})} = \left(\int_{\mathbb{Y}|^{c_1}_{\rm Dir}} q^{(5+1)} \right) \mathrm{e}^{-i2\pi\hat{\eta}_{\rm DF}(\mathbb{Y}|_{\rm Dir})}$$

$$\nabla_\eta \equiv \delta_\eta + i\mathcal{L}_\eta$$

$$-i\mathcal{L}_\eta = \sum_i v_i(x)^\dagger \delta_\eta v_i(x)$$

$$\int_{\mathbb{Y}|^{c_1}_{\rm Dir}} q^{(5+1)} \equiv \sum_{x,t\in c_1} q^{(5+1)}(y,s)\big|_{s=1}$$

(Integral of topological field on 5+1, 6 dim. lattices)

$$q^{(5+1)}(y,s) \equiv \mathrm{Im}\,\mathrm{Tr}\{D_s(U_s)\, U_s^{-1}(y,\mu)\, J_{\rm w}^{(5)}(y,\mu)|_{U=U_s}\}$$

$$q^{(5+1)}(y,s)\big|_{s=1} = \lim_{a_6\to 0}\lim_{N_6\to\infty} q^{(6)}(z)\big|_{\mathbb{Y}^{c_1 c_2^{-1}}_{\rm AP}}$$

η-invariant on the lattice

$$i\not{D}^{(5)}\psi_\lambda = \lambda\psi_\lambda$$

$$\eta_Y^{(5)} \equiv \frac{1}{2}\Big(\sum_{\lambda\neq 0}\mathrm{sign}(\lambda) + \ \mathrm{dim\ ker}\ (i\not{D}^{(5)})\Big)$$

APS Index Theorem on the lattice

[*Pedersen-YK (2019)*]
cf. [*Aoyama-YK (1999)*]

- **APS Index**

$$I(\mathbb{Z}|_{\rm Dir}) \equiv -\frac{1}{2}\mathrm{Tr}\left\{ \left. \frac{H_{\rm w}^{(6)}}{\sqrt{{H_{\rm w}^{(6)}}^2}} \right|_{\rm Dir} \right\}$$

[*Kawai, Fukaya et al (2019)*]

$$(-1)^{I(\mathbb{Z}|_{\rm Dir})} = \frac{\det X_{\rm w}^{(6)}|_{\rm Dir}}{\left|\det X_{\rm w}^{(6)}|_{\rm Dir}\right|} \qquad I(\mathbb{Z}|_{\rm Dir}) = \mathrm{Index}\, D_{\rm ov}^{(6)}|_{\rm Dir}$$

cf. [*Witten (2016)*]

- **APS Index Theorem**

$$(-1)^{I(\mathbb{Z}|_{\rm Dir})} = \mathrm{e}^{i\pi P(\mathbb{Z}|^{c}_{\rm APS})} \quad \mathrm{e}^{i\pi\left[\hat{\eta}'(\mathbb{Y}_1|_{\rm AP}) - \hat{\eta}'(\mathbb{Y}_0|_{\rm AP})\right]}$$

$$P(\mathbb{Z}|^{c_1}_{\rm APS}) = \lim_{N\to\infty} \sum_{y,s\in c} q^{(6)}(z)$$

J. Preskill, e-Print: 1811.10085 [hep-lat], PoS LATTICE2018 (2018) 024

"Simulating quantum field theory with a quantum computer"

" • Chiral fermions pose another important challenge. The standard model is chiral; that is, for both quarks and leptons, the massless left-handed and right-handed fermions carry different gauge charges. Yet existing methods for formulating fermion theories on the lattice always yield nonchiral theories — e.g., if we try to introduce left-handed fermions with a specified charge we also get unwanted right-handed particles with the same charge. We've been facing this problem for over 40 years, but there is still no accepted method for regulating a chiral theory. That's embarrassing."

J. Preskill, e-Print: 1811.10085 [hep-lat], PoS LATTICE2018 (2018) 024

"Simulating quantum field theory with a quantum computer"

"This long-standing problem may be nearing a resolution, guided in part by recent insights regarding symmetry-protected topological phases of matter. Two old ideas are:

(1) To realize a D-dimensional chiral theory on the lattice, we can introduce an extra spatial dimension, so that the left-handed and right-handed fermions live on two different D-dimensional edges of a (D + 1)- dimensional bulk [45].

(2) To realize a D-dimensional chiral theory on the lattice, we can introduce strong interactions for the express purpose of removing the unwanted right-handed fermions (by giving them large masses) while preserving the massless left-handed fermions [46].

It seems likely that (1) and (2) together work more effectively than either (1) or (2) by itself [47]. That's because separating the two edges with a higher-dimensional bulk makes it easier to apply the strong interactions to one chirality without affecting the other.

The efficacy of this method still needs to be demonstrated convincingly, but if it works that will settle the longstanding open question whether quantum field theories with chiral fermions really exist, and will also open the door for classical and quantum studies of the rich dynamics of strongly-coupled chiral gauge theories, with potential applications to physics beyond the standard model."

Let me add …

In view of the fact that "the overlap fermion is nothing but the low energy effective (local & lattice) theory for the edge chiral modes of DWF infinitely separated", the mirror fermion models with overlap fermions (1) and (refined) Eichten-Presskill interaction terms (2) are the simplest-possible effective framework to construct & test CGT on the lattice !

cf. [*Y.K. (2017)*]

“Eventually, all things merge into one and a river runs through it”

What is the sound of one hand clapping?

隻手音声

両手の鳴る音は知る。
片手の鳴る音はいかに？
ー 禅の公案 ー